

STABILITY OF HARMONIC MAPS AND EIGENVALUES OF THE LAPLACIAN

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Dedicated to Professor Ichiro Satake on his sixtieth birthday

ABSTRACT. The index and nullity of the Hessian of the energy for every harmonic map are estimated above by a geometric quantity. The stability theory of harmonic maps is developed and as an application, the Kähler version of the Lichnerowicz-Obata theorem about the first eigenvalue of the Laplacian is proved.

0. Introduction and statement of results. In this paper, we deal with the Hessian (the so-called Jacobi operator) of the energy for a harmonic map between two Riemannian manifolds, as the analogue of Morse theory for geodesics. The Morse theory for geodesics determines the homotopy type of the path space, using the notion of the index and nullity of the Hessian of the length of a geodesic. The celebrated Morse index theorem tells us that the index or nullity of a geodesic coincides with the number of conjugate points along the geodesic, and this gives an upper estimate for the index and nullity in terms of the length of the geodesic under a curvature assumption, giving a result known as the Morse-Schoenberg theorem:

THEOREM (MORSE - SCHOENBERG [G.K.M]). *Assume that the sectional curvature ${}^N K$ of a Riemannian manifold (N, h) is bounded above by a positive constant ${}^N K \leq a$. Then the nullity and index of a geodesic $\gamma; [0, 2\pi] \rightarrow (N, h)$ satisfy*

$$\text{Index}(\gamma) + \text{Nullity}(\gamma) \leq (n-1) \left[L \frac{\sqrt{a}}{\pi} \right],$$

where L is the length of the geodesic γ and $[x]$ is the integer part of a positive real number x .

Harmonic maps are the natural extensions of the notion of geodesic. They are defined as critical points of the energy on the space of smooth maps between two Riemannian manifolds (cf. [E.S, E.L]). Therefore it is reasonable to look for the analogue of Morse theory for harmonic maps. In order to do this we have to deal with the index and nullity of the Hessian of the energy and investigate their quantitative behavior for a general harmonic map.

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The first aim of this paper is to extend the above Morse-Schoenberg theorem to a general harmonic map, although we do not yet know the index theorem for harmonic maps. Namely we will give a general upper estimate on the index and nullity for every harmonic map in terms of a geometric quantity which coincides with the length in case of a geodesic (cf. Corollary 3.5). Note that the Morse-Schoenberg theorem gives a stability theorem for a geodesic with a small length, i.e., the index and nullity vanish for a geodesic with a small length. Our upper estimate also gives a stability theorem (cf. Corollary 3.3) for a harmonic map with the above quantity small, which generalizes the stability theorem for a minimal immersion obtained by D. Hoffman [H], H. Mori [M], and S. Tanno [T2].

Now we call a harmonic map ϕ from a compact Riemannian manifold *without boundary* stable if the index of ϕ is zero (cf. [E.L]). We will study stable harmonic maps to determine what kind of harmonic maps are stable. Moreover, the second aim of this paper is to give an application of the stability of a harmonic map. It is well known that a holomorphic map between two compact Kähler manifolds is a stable harmonic map. In fact, it minimizes the energy within its homotopy class. In particular, the identity map of a compact Kähler manifold is stable. Using this fact, we give a Kähler version of Lichnerowicz-Obata's theorem about the lower estimate on the first eigenvalue of the Laplacian. More precisely we give

THEOREM 4.2. *Let (M, g) be a compact Kähler manifold whose Ricci curvature Ric_M is bounded below by a positive constant $\alpha > 0$. Then the first eigenvalue $\lambda_1(M, g)$ of the Laplacian satisfies $\lambda_1(M, g) \geq 2\alpha$. If the equality holds, then M admits a nonzero holomorphic vector field.*

The third goal of this paper is to investigate a special kind of stable harmonic map. Y. L. Xin [X] gave a remarkable result that each nonconstant harmonic map from the canonical unit sphere S^n , $n \geq 3$, into another Riemannian manifold is unstable. It is natural to ask the following:

Does there exist a deformation g_t , $0 < t < \infty$, of the standard metric g_1 on S^n such that if g_t is far from g_1 , then (S^n, g_t) admits a stable harmonic map?

In order to answer this question, we investigate the stability of the projection of the Riemannian submersion with totally geodesic fibers since the projection is a typical example of a harmonic map. Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers. Following [B.B], we consider the canonical variation g_t , $0 < t < \infty$, of the metric g which also gives a Riemannian submersion $\phi; (M, g_t) \rightarrow (N, h)$ with totally geodesic fibers. Then we have:

THEOREM 7.3. *Assume that the identity map of (N, h) is stable. Then there exists a small number ε such that for every $0 < t < \varepsilon$, the Riemannian submersion $\phi; (M, g_t) \rightarrow (N, h)$ is stable.*

Since the identity map of the complex projective space $(\mathbb{C}P^n, h)$ is stable, the Hopf fibering $\pi; (S^{2n+1}, g_t) \rightarrow (\mathbb{C}P^n, h)$ is stable for $0 < t < \varepsilon$, for the canonical variation g_t , $0 < t < \infty$, with $g_1 =$ canonical metric. This gives an example which contrasts with the instability theorem of Xin.

Finally, we investigate homogeneous Riemannian submersions and calculate the index and nullity of the Hopf fibering (cf. Corollary 8.12).

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CHAPTER I. THE INDEX AND NULLITY OF A GENERAL HARMONIC MAP

1. Preliminaries.

1.1. In this section, following [E.L], we describe the second variational formula of the energy functional obtained in [Ma, Sm].

Let (M, g) and (N, h) be Riemannian manifolds of dimension m and n respectively. Let $\phi; M \rightarrow N$ be a smooth map. Let $E = \phi^{-1}TN$ be the bundle induced by ϕ over M from the tangent bundle TN of N . We denote by $\Gamma(E)$, the space of all sections V of E , that is, $V \in \Gamma(E)$ implies that V is a map of M into E such that $V_x \in T_{\phi(x)}N$ for all $x \in M$. For $X \in \Gamma(TM)$, we define $\phi_*X \in \Gamma(E)$ by $(\phi_*X)_x := \phi_{*x}X_x \in T_{\phi(x)}N$, $x \in M$, where ϕ_{*x} is the differential of ϕ at x . For $Y \in \Gamma(TN)$, we also define $\tilde{Y} \in \Gamma(E)$ by $\tilde{Y}_x := Y_{\phi(x)}$, $x \in M$.

We denote by ∇ and ${}^N\nabla$ the Levi-Civita connections of (M, g) and (N, h) respectively. Then we give the induced connection $\tilde{\nabla}$ on E by

$$(1.1) \quad \tilde{\nabla}_X V := {}^N\nabla_{\phi_*X} V,$$

for a tangent vector X in M .

We define the *tension field* $\tau(\phi) \in \Gamma(E)$ of ϕ by

$$\tau(\phi) := \sum_{i=1}^m (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i),$$

where $\{e_i\}_{i=1}^m$ is a (locally defined) orthonormal frame field on M . We call ϕ *harmonic* if $\tau(\phi) = 0$. For a relatively compact domain Ω in M , the *energy* $E(\Omega, \phi)$ of ϕ on Ω is defined by

$$E(\Omega, \phi) := \int_{\Omega} e(\phi)(x) * 1,$$

where $e(\phi)(x) := \frac{1}{2} \sum_{i=1}^m h(\phi_* e_i, \phi_* e_i)$ and $*1$ is the volume element of (M, g) . We denote $E(\phi) := E(M, \phi)$ when defined. For an element V in $\Gamma(E)$, let $\phi_t; M \rightarrow N$ be a one-parameter family of maps from M into N with $\phi_0 = \phi$, and $d\phi_t(x)/dt|_{t=0} = V_x$, $x \in M$. If $V \in \Gamma(E)$ has compact support, it is known (cf. [E.S, E.L, Ma]) that

$$(1.2) \quad \left. \frac{d}{dt} E(\phi_t) \right|_{t=0} = - \int_M h(V, \tau(\phi)) * 1.$$

Moreover, if $\phi; (M, g) \rightarrow (N, h)$ is harmonic and $V \in \Gamma(E)$ has compact support, then

$$(1.3) \quad \left. \frac{d^2}{dt^2} E(\phi_t) \right|_{t=0} = \int_M h(V, J_{\phi} V) * 1,$$

where the operator $J_\phi; \Gamma(E) \rightarrow \Gamma(E)$, called the *Jacobi operator* of ϕ , is a second order elliptic differential operator given by

$$(1.4) \quad J_\phi V := - \sum_{i=1}^m \left\{ \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} V - \tilde{\nabla}_{\nabla_{e_i} e_i} V \right\} - \sum_{i=1}^m {}^N R(\phi_* e_i, V) \phi_* e_i,$$

for $V \in \Gamma(E)$. Here ${}^N R$ is the curvature tensor of (N, h) given by

$$(1.5) \quad {}^N R(X, Y)Z := {}^N \nabla_{[X, Y]} Z - {}^N \nabla_X {}^N \nabla_Y Z + {}^N \nabla_Y {}^N \nabla_X Z,$$

for $X, Y, Z \in \Gamma(TN)$.

For a relatively compact domain Ω in M , let us consider the Dirichlet eigenvalue problem of J_ϕ

$$(1.6) \quad \begin{cases} J_\phi V = \lambda V & \text{on } \Omega, \\ V = 0 & \text{on } \partial\Omega. \end{cases}$$

If M is a closed manifold, we consider the eigenvalue problem of J_ϕ

$$(1.7) \quad J_\phi V = \lambda V, \quad V \in \Gamma(E).$$

It is known that the spectra of both problems (1.6) and (1.7) consist of discrete eigenvalues with finite multiplicities. The *index* of ϕ on Ω , denoted by $\text{Index}_\Omega(\phi)$, is defined as the sum of the multiplicities of the eigenvalues of the problem (1.6), and the *index* of ϕ , denoted by $\text{Index}(\phi)$, is defined as the sum of the multiplicities of the eigenvalues of (1.7) when M is a closed manifold. The dimension of the zero eigenspace of (1.6) (resp. (1.7)) is called the *nullity* of ϕ on Ω (resp. the *nullity* of ϕ), denoted by $\text{Nullity}_\Omega(\phi)$ (resp. $\text{Nullity}(\phi)$). The harmonic map $\phi; (M, g) \rightarrow (N, h)$ is *stable* (resp. *stable* on Ω) if $\text{Index}(\phi) = 0$ (resp. $\text{Index}_\Omega(\phi) = 0$).

1.2. To estimate the index and nullity of a harmonic map, we introduce the quantity ${}^N R^\phi$ or ${}^N R_\Omega^\phi$ as follows.

DEFINITION 1.1. For a smooth map $\phi; (M, g) \rightarrow (N, h)$, define ${}^N R^\phi$ by

$$(1.8) \quad {}^N R^\phi := \sup_{x \in M} \sup_{v \in T_{\phi(x)} N} \sum_{i=1}^m \frac{h({}^N R(\phi_* e_i, v) \phi_* e_i, v)}{h(v, v)}.$$

For a relatively compact domain Ω in M , define ${}^N R_\Omega^\phi$ by

$$(1.9) \quad {}^N R_\Omega^\phi := \sup_{x \in \Omega} \sup_{v \in T_{\phi(x)} N} \sum_{i=1}^m \frac{h({}^N R(\phi_* e_i, v) \phi_* e_i, v)}{h(v, v)}.$$

Note that these quantities do not depend on the choice of $\{e_i\}_{i=1}^m$. We have immediately

LEMMA 1.2. Assume that the sectional curvature ${}^N K$ of (N, h) is bounded above by a positive constant a so that ${}^N K(\pi) \leq a$ for all planes π in $T_y N$, $y \in N$. Then we have

$$(1.10) \quad {}^N R^\phi \leq 2aE^\infty(\phi),$$

$$(1.11) \quad {}^N R_\Omega^\phi \leq 2aE^\infty(\Omega, \phi).$$

Here $E^\infty(\phi) := \sup_{x \in M} e(\phi)(x)$ and $E^\infty(\Omega, \phi) := \sup_{x \in \Omega} e(\phi)(x)$.

In fact, it is obvious that

$$\sum_{i=1}^m h({}^N R(\phi_* e_i, v) \phi_* e_i, v) \leq a \left\{ \sum_{i=1}^m h(\phi_* e_i, \phi_* e_i) \right\} h(v, v)$$

at each point of M . Note that

$$E(\Omega, \phi) \leq E^\infty(\Omega, \phi) \text{Vol } \Omega \quad \text{and} \quad E(\phi) \leq E^\infty(\phi) \text{Vol } M \quad \text{if } \text{Vol } M < \infty.$$

EXAMPLE 1.3. Let $\phi; (M, g) \rightarrow (N, h)$ be an isometric immersion. Then $e(\phi)(x) = m/2$ at each point. Therefore

$$(1.12) \quad E^\infty(\phi) = E^\infty(\Omega, \phi) = m/2,$$

$$(1.12') \quad {}^N R_\Omega^\phi \leq {}^N R^\phi \leq ma,$$

for every relatively compact domain Ω in M . In particular, let $\phi; [0, 2\pi] \rightarrow (N, h)$ be a geodesic with length L . Then

$$(1.13) \quad E^\infty(\phi) = L^2/8\pi^2.$$

EXAMPLE 1.4. Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion (cf. §6). Then we can choose an orthonormal local frame $\{e_i\}_{i=1}^m$ on M such that $\phi_* e_i = e'_i$, $1 \leq i \leq n$, and $\phi_* e_i = 0$, $n+1 \leq i \leq m$, where $m = \dim M$, $n = \dim N$, and $\{e'_i\}_{i=1}^n$ is an orthonormal local frame on N . Then the Ricci curvature of (N, h) , $\text{Ric}_N(v)$, $v \in T_{\phi(x)}N$, is by definition $\sum_{i=1}^n h({}^N R(\phi_* e_i, v) \phi_* e_i, v)/h(v, v)$. Therefore, since ϕ is surjective, we have

$$(1.14) \quad {}^N R^\phi = \sup_N \text{Ric}_N \quad \text{and} \quad {}^N R_\Omega^\phi = \sup_{\phi(\Omega)} \text{Ric}_N.$$

2. An estimate for the index and nullity of a harmonic map. At first, let us recall a method of Berard and Gallot (cf. [B.G]) to give estimates of the Betti number and dimension of the moduli space of Einstein metrics, and the dimension of harmonic spinors. In this section, we point out that their method works well in the case of the Dirichlet eigenvalue problem for a relatively compact domain Ω in a complete Riemannian manifold (M, g) .

Let (M, g) be a complete Riemannian manifold of dimension m , and E , a vector bundle over M with an inner product $\langle \cdot, \cdot \rangle$ and a connection $\tilde{\nabla}$ compatible with $\langle \cdot, \cdot \rangle$, that is,

$$\nabla_X \langle s, s' \rangle = \langle \tilde{\nabla}_X s, s' \rangle + \langle s, \tilde{\nabla}_X s' \rangle, \quad X \in \Gamma(TM), \quad s, s' \in \Gamma(E).$$

Then we can define the rough Laplacian $\bar{\Delta}$ on E so that

$$(2.1) \quad \bar{\Delta}s := \sum_{i=1}^m \left\{ \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} s - \tilde{\nabla}_{\nabla_{e_i} e_i} s \right\}, \quad s \in \Gamma(E),$$

where $\{e_i\}_{i=1}^m$ is an orthonormal local frame field on M .

Let $\bar{\lambda}_1(\Omega) \leq \bar{\lambda}_2(\Omega) \leq \cdots \leq \bar{\lambda}_i(\Omega) \leq \cdots$ be the spectrum of the Dirichlet eigenvalue problem of the rough Laplacian $\bar{\Delta}$ (2.1) of the vector bundle E over a relatively compact domain Ω in M :

$$\begin{cases} -\bar{\Delta}s = \lambda s & \text{on } \Omega, \\ s = 0 & \text{on } \partial\Omega, \end{cases}$$

where s is a section of E on the closure $\bar{\Omega}$ of Ω . Consider the zeta function $\bar{Z}_{E,\Omega}(t)$ defined by

$$\bar{Z}_{E,\Omega}(t) := \sum_{i=1}^{\infty} e^{-t\bar{\lambda}_i(\Omega)}, \quad t > 0.$$

Similarly, let $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_i(\Omega) \leq \cdots$ be the spectrum of the Dirichlet eigenvalue problem of the Laplace-Beltrami operator Δ_M for the domain Ω , and $Z_{\Omega}(t)$ be the zeta function defined by

$$(2.2) \quad Z_{\Omega}(t) := \sum_{i=1}^{\infty} e^{-t\lambda_i(\Omega)}, \quad t > 0.$$

Then we have the analogue of a theorem of Hess, Schrader and Uhlenbrock:

THEOREM 2.1. *If l is the rank of E then*

$$(2.3) \quad \bar{Z}_{E,\Omega}(t) \leq l Z_{\Omega}(t), \quad t > 0.$$

PROOF. This can be proved as in [B.G]. Assume that $s(t, x) \in E_x$, $t > 0$, $x \in \bar{\Omega}$, satisfies the heat equation with the Dirichlet boundary condition

$$\begin{cases} \left(\frac{\partial}{\partial t} - \bar{\Delta} \right) s(t, x) = 0 & \text{on } (0, \infty) \times \Omega, \\ s(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases}$$

For each $\varepsilon > 0$, let $f_{\varepsilon} := (|s|^2 + \varepsilon^2)^{1/2}$ on $(0, \infty) \times \bar{\Omega}$. Then it can be proved as in [H.S.U.] that $\langle -\bar{\Delta}s, s \rangle \leq f_{\varepsilon}(-\Delta_M f_{\varepsilon})$ on $(0, \infty) \times \Omega$. Therefore f_{ε} satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_M \right) f_{\varepsilon} \leq 0 \quad \text{on } (0, \infty) \times \Omega.$$

Then we can apply f_{ε} to the following maximum principle for the heat kernel:

THEOREM (MAXIMUM PRINCIPLE). *Let Ω be a relatively compact domain in M , and let $0 < T < \infty$. Assume that u is a real valued continuous function on $[0, T] \times \Omega$ and satisfies the inequality*

$$\frac{\partial}{\partial t} u - \Delta_M u \leq 0 \quad \text{on } (0, T) \times \Omega.$$

Then u attains its maximum on the set $\{0\} \times \Omega$ or $[0, T] \times \partial\Omega$.

For proof, see [F, p. 204].

If $f_{\varepsilon}(0, x) \leq f(0, x) + \varepsilon$, then $f_{\varepsilon}(t, x) \leq f(t, x) + \varepsilon$. Hence for every integrable section s of E on $\bar{\Omega}$ with the Dirichlet condition $s = 0$ on $\partial\Omega$, we have

$$(2.4) \quad |(e^{t\bar{\Delta}}s)(x)| \leq (e^{t\Delta_M}|s|)(x).$$

Therefore applying $s(z) = \sum_{i=1}^l \delta_{z,y} u_j(z)$ to (2.4), where $\delta_{z,y}$ is the Dirac function at y and $\{u_j(z)\}_{j=1}^l$ is an orthonormal basis of the fiber E_z at each point z in M , and noting that $|s(z)| = l\delta_{z,y}$, we have the desired inequality (2.3). Q.E.D.

We denote the spectrum of the Dirichlet eigenvalue problem of J_{ϕ} on Ω by

$$(2.5) \quad \tilde{\lambda}_1(\Omega) \leq \tilde{\lambda}_2(\Omega) \leq \cdots \leq \tilde{\lambda}_i(\Omega) \leq \cdots$$

and define $\tilde{Z}_\Omega(t) := \sum_{i=1}^\infty e^{-t\tilde{\lambda}_i(\Omega)}$. Then we have

PROPOSITION 2.2. *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g) . Let $\phi; (M, g) \rightarrow (N, h)$ be a harmonic map. Then*

$$(2.6) \quad \text{Index}_\Omega(\phi) + \text{Nullity}_\Omega(\phi) \leq \tilde{Z}_\Omega(t) \leq n \inf \left\{ e^{t^N R_\Omega^\phi} Z_\Omega(t); 0 < t < \infty \right\},$$

where $n = \dim N$, $^N R_\Omega^\phi$ is defined in §1, and $Z_\Omega(t)$ is the zeta function of the Dirichlet eigenvalue of Δ_M on Ω defined by (2.2).

3. The index and the nullity of a harmonic map from a domain.

3.1. We retain the notation of §2. We have

THEOREM 3.1. *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g) , with $\phi; (M, g) \rightarrow (N, h)$ a harmonic map of (M, g) into an arbitrary Riemannian manifold (N, h) of dimension n . Then*

(i) $\lambda_1(\Omega) \geq ^N R_\Omega^\phi \Rightarrow \text{Index}_\Omega(\phi) = 0$ and $\text{Nullity}_\Omega(\phi) \leq n$,

(ii) $\lambda_1(\Omega) > ^N R_\Omega^\phi \Rightarrow \text{Index}_\Omega(\phi) = \text{Nullity}_\Omega(\phi) = 0$.

That is, if $\lambda_1(\Omega) \geq ^N R_\Omega^\phi$, then the harmonic map $\phi; (M, g) \rightarrow (N, h)$ is stable on Ω .

PROOF. By Proposition 2.2, the zeta function $\tilde{Z}_\Omega(t) = \sum_{i=1}^\infty e^{-t\tilde{\lambda}_i(\Omega)}$ of J_ϕ on Ω satisfies

$$\tilde{Z}_\Omega(t) \leq n e^{t^N R_\Omega^\phi} Z_\Omega(t) = n e^{t(^N R_\Omega^\phi - \lambda_1(\Omega))} \left\{ 1 + \sum_{i=2}^\infty e^{t(\lambda_1(\Omega) - \lambda_i(\Omega))} \right\},$$

where $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_i(\Omega) \leq \dots$ is the spectrum of the Dirichlet eigenvalue problem of the Laplace-Beltrami operator Δ_M on Ω . Noting the fact that $\lambda_i(\Omega) > \lambda_1(\Omega)$, $i = 2, 3, \dots$, the assumption $^N R_\Omega^\phi \leq \lambda_1(\Omega)$ implies that the limit of the right-hand side of the above inequality is less than or equal to n when $t \rightarrow \infty$. Then $\text{Index}_\Omega(\phi) = 0$ and $\text{Nullity}_\Omega(\phi) \leq n$. If $^N R_\Omega^\phi < \lambda_1(\Omega)$, the limit of the right-hand side of the inequality is zero when $t \rightarrow \infty$. Therefore $\text{Index}_\Omega(\phi) = \text{Nullity}_\Omega(\phi) = 0$. Q.E.D.

COROLLARY 3.2. *Let $B_r(0)$ be a geodesic ball with radius r whose center is a certain point o in the m -dimensional standard unit sphere (S^m, can) of constant curvature one. We choose the radius r with $0 < r < \pi/2$ in such a way that $\lambda_1(B_r(0)) = m - 1$. Then, for every domain Ω in S^m whose volume $\text{Vol}(\Omega)$ is less than or equal to the volume $\text{Vol}(B_r(0))$, the identity map $\text{id}; (S^m, \text{can}) \rightarrow (S^m, \text{can})$ is stable on Ω .*

PROOF. By Example 1.4, we have $^N R_\Omega^\phi = m - 1$ for every domain Ω in S^m . In this case, Theorem 3.1 implies that, if $\lambda_1(\Omega) \geq m - 1$, then the identity map $\phi = \text{id}; (S^m, \text{can}) \rightarrow (S^m, \text{can})$ is stable on Ω . By a theorem of P. Bérard and D. Meyer (cf. [B.M]), if $\text{Vol}(\Omega) \leq \text{Vol}(B_r(0))$, then $\lambda_1(\Omega) \geq \lambda_1(B_r(0)) = m - 1$. Q.E.D.

It is known (cf. [C.L, B.G, U2]) that there exists a positive constant $C(M, g)$ depending only on (M, g) such that the eigenvalues $\lambda_i(\Omega)$ of the Dirichlet eigenvalue problem of the Laplace-Beltrami operator Δ_M on the domain Ω satisfy

$$(3.1) \quad \lambda_i(\Omega) \geq C(M, g) \text{Vol}(\Omega)^{-2/m} i^{2/m}, \quad i = 1, 2, \dots,$$

where $m = \dim M$. In particular,

$$(3.2) \quad \lambda_1(\Omega) \geq C(M, g) \operatorname{Vol}(\Omega)^{-2/m}.$$

Thus Theorem 3.1 implies

COROLLARY 3.3. *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g) , with $\phi; (M, g) \rightarrow (N, h)$ a harmonic map. Then*

$$C(M, g) \operatorname{Vol}(\Omega)^{-2/m} \geq {}^N R_\Omega^\phi \Rightarrow \phi \text{ is stable on } \Omega.$$

In particular, assume that the sectional curvature ${}^N K$ of (N, h) is bounded above by a positive constant ${}^N K \leq a$. Then

$$C(M, g) \operatorname{Vol}(\Omega)^{-2/m} \geq 2aE^\infty(\Omega, \phi) \Rightarrow \phi \text{ is stable on } \Omega.$$

If Ω is “small” in (M, g) , then $\operatorname{Vol}(\Omega)^{-2/m}$ tends to infinity and ${}^N R_\Omega^\phi$ still remains bounded. Therefore Corollary 3.3 implies that a harmonic map $\phi; (M, g) \rightarrow (N, h)$ is stable on a “sufficiently small” domain Ω in M .

3.2. In this section, we estimate $\operatorname{Index}_\Omega(\phi)$ and $\operatorname{Nullity}_\Omega(\phi)$. By Proposition 2.2 and (3.1), we have

$$\begin{aligned} \operatorname{Index}_\Omega(\phi) + \operatorname{Nullity}_\Omega(\phi) &\leq n \inf \left\{ e^{t {}^N R_\Omega^\phi} Z_\Omega(t); 0 < t < \infty \right\} \\ &\leq n \inf \left\{ e^{t {}^N R_\Omega^\phi} \sum_{k=1}^{\infty} e^{-t C(M, g) \operatorname{Vol}(\Omega)^{-2/m} k^{2/m}}; 0 < t < \infty \right\} \\ &\leq n \inf \left\{ e^{at/b} \sum_{k=1}^{\infty} e^{-tk^{2/m}}; 0 < t < \infty \right\}, \end{aligned}$$

where we put $m = \dim M$, $n = \dim N$, $a := {}^N R_\Omega^\phi$, and $b := C(M, g) \operatorname{Vol}(\Omega)^{-2/m}$. In case $a \leq b$, we have Corollary 3.3. So assume $a > b$. We put $a/b = 1 + D$, $D > 0$. We may write

$$(3.3) \quad e^{at/b} \sum_{k=1}^{\infty} e^{-tk^{2/m}} = e^{(a/b-1)t} \sum_{k=1}^{\infty} e^{-(k^{2/m}-1)t}.$$

In case $m = 1, 2$, the right-hand side of (3.3)

$$\leq e^{(a/b-1)t} \sum_{k=0}^{\infty} e^{-tk} = e^{(a/b-1)t} (1 - e^{-t})^{-1}.$$

Putting $e^t = 1 + 1/D$, we have

$$\inf \left\{ e^{at/b} \sum_{k=1}^{\infty} e^{-tk^{2/m}}; 0 < t < \infty \right\} \leq \left(1 + \frac{1}{D} \right)^D (1 + D).$$

In case $m \geq 3$,

$$\begin{aligned} \sum_{k=1}^{\infty} e^{-t(k^{2/m}-1)} &= 1 + e^t \sum_{k=2}^{\infty} e^{-tk^{2/m}} \\ &\leq 1 + e^t \int_1^{\infty} e^{-tx^{2/m}} dx \\ &= 1 + \frac{m}{2} t^{-m/2} \int_t^{\infty} z^{m/2-1} e^{-z} dz \\ &\leq \begin{cases} 1 + \frac{m}{2} t^{-m/2} p! e^{-t} \sum_{k=0}^p \frac{t^k}{k!}, & \text{if } m = 2(p+1), p \geq 1, \\ 1 + \frac{m}{2} t^{-(m+1)/2} p! e^{-t} \sum_{k=0}^p \frac{t^k}{k!}, & \text{if } m = 2p+1, p \geq 1. \end{cases} \end{aligned}$$

Putting $e^t = 1 + 1/D$, we have

$$\begin{aligned} &\inf \left\{ e^{at/b} \sum_{k=1}^{\infty} e^{-tk^{2/m}}; 0 < t < \infty \right\} \\ &\leq \begin{cases} \left(1 + \frac{1}{D}\right)^D \{1 + P(D)\}, & \text{if } m = 2(p+1), p \geq 1, \\ \left(1 + \frac{1}{D}\right)^D \{1 + Q(D)\}, & \text{if } m = 2p+1, p \geq 1, \end{cases} \end{aligned}$$

where

$$(3.4) \quad P(D) := (p+1)! \sum_{k=0}^p \frac{1}{k!} \left\{ \frac{1}{\log(1 + \frac{1}{D})} \right\}^{p+1-k}, \quad \text{if } m = 2(p+1), p \geq 1,$$

$$(3.5) \quad Q(D) := \frac{m}{2} p! \sum_{k=0}^p \frac{1}{k!} \left\{ \frac{1}{\log(1 + \frac{1}{D})} \right\}^{p+1-k}, \quad \text{if } m = 2p+1, p \geq 1.$$

We can give another estimate of $\text{Index}_{\Omega}(\phi)$ and $\text{Nullity}_{\Omega}(\phi)$. In fact, we have

$$\sum_{k=1}^{\infty} e^{-tk^{2/m}} \leq \int_0^{\infty} e^{-tx^{2/m}} dx = \Gamma\left(\frac{m}{2} + 1\right) t^{-m/2}.$$

Therefore we obtain

$$\inf \left\{ e^{at/b} \sum_{k=1}^{\infty} e^{-tk^{2/m}}; 0 < t < \infty \right\} \leq \frac{\Gamma(m/2 + 1) e^{m/2}}{(m/2)^{m/2}} \left(\frac{a}{b}\right)^{m/2}.$$

Summing up, we obtain

THEOREM 3.4. *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g) , and $\phi; (M, g) \rightarrow (N, h)$, a harmonic map. Then $\text{Index}_{\Omega}(\phi)$ and $\text{Nullity}_{\Omega}(\phi)$ are estimated in terms of the quantity $D := {}^N R_{\Omega}^{\phi} C(M, g)^{-1} \text{Vol}(\Omega)^{2/m} - 1$ as follows:*

(i) For $m = 1, 2$,

$$\text{Index}_{\Omega}(\phi) + \text{Nullity}_{\Omega}(\phi) \leq n(1 + 1/D)^D \{1 + D\}.$$

(ii) For $m = 2(p+1)$, $p \geq 1$,

$$\text{Index}_{\Omega}(\phi) + \text{Nullity}_{\Omega}(\phi) \leq n(1 + 1/D)^D \{1 + P(D)\}.$$

(iii) For $m = 2p + 1$, $p \geq 1$,

$$\text{Index}_{\Omega}(\phi) + \text{Nullity}_{\Omega}(\phi) \leq n(1 + 1/D)^D \{1 + Q(D)\}.$$

(iv) For $m \geq 1$,

$$\text{Index}_{\Omega}(\phi) + \text{Nullity}_{\Omega}(\phi) \leq n \frac{\Gamma(m/2 + 1)e^{m/2}}{(m/2)^{m/2}} (1 + D)^{m/2},$$

where $P(D)$ and $Q(D)$ are the functions of D given by (3.4), (3.5), respectively, and $m = \dim M$, $n = \dim N$.

REMARK. Since the function

$$f(D) = \frac{1}{\log(1 + 1/D)}$$

of D satisfies $f(D) \rightarrow 0$ as $D \rightarrow 0$ and $f(D) \sim D$ as $D \rightarrow \infty$, the functions $P(D)$ and $Q(D)$ satisfy $\lim_{D \rightarrow 0} P(D) = \lim_{D \rightarrow 0} Q(D) = 0$, and $P(D) \sim (m/2)!D^{m/2}$, and $Q(D) \sim (m/2)((m-1)/2)!D^{(m+1)/2}$ as $D \rightarrow \infty$.

Using (1.11), we obtain

COROLLARY 3.5. Assume that the sectional curvature ${}^N K$ of (N, h) is bounded above by a positive constant a . Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g) , and $\phi; (M, g) \rightarrow (N, h)$ a harmonic map. Then

$$\text{Index}_{\Omega}(\phi) + \text{Nullity}_{\Omega}(\phi) \leq n\Gamma\left(\frac{m}{2} + 1\right) \left\{ \frac{C(M, g)^{-1}ea}{m} \right\}^{m/2} E^{\infty}(\Omega, \phi)^{m/2} \text{Vol}(\Omega).$$

In particular, in the case $(M, g) = (\mathbf{R}^m, \text{can})$, the standard Euclidean space, since $C(\mathbf{R}^m, \text{can}) = 4\pi^2\omega_m^{-2/m}$ (cf. [U2]) with $\omega_m = \pi^{m/2}/\Gamma(m/2 + 1)$, the volume of the unit ball, we have, for every harmonic map $\phi; (\mathbf{R}^m, \text{can}) \supset \Omega \rightarrow (N, h)$,

$$\text{Index}_{\Omega}(\phi) + \text{Nullity}_{\Omega}(\phi) \leq n \left(\frac{ea}{m\pi} \right)^{m/2} E^{\infty}(\Omega, \phi)^{m/2} \text{Vol}(\Omega).$$

REMARK. It seems that the index and nullity of harmonic maps might be estimated above in terms of the quantity $\int_{\Omega} e(\phi)^{m/2} * 1$.

In the case of a closed manifold M , we get the following

THEOREM 3.6. Let (M, g) be a closed Riemannian manifold of dimension $m \geq 2$ whose Ricci curvature Ric_M is bounded below by a positive constant $(m-1)\delta > 0$. Let $\phi; (M, g) \rightarrow (N, h)$ be a harmonic map of (M, g) into an arbitrary Riemannian manifold (N, h) of dimension n .

(i) In case $m \geq 3$,

$$\text{Index}(\phi) + \text{Nullity}(\phi) \leq n \left(1 + \frac{1}{A} \right)^A \{1 + (m-1)!m^{m-1}A(1+A)^{m-1}\},$$

where $A := {}^N R^{\phi}/m\delta$ and ${}^N R^{\phi}$ is the quantity in §1.

(ii) In case $m = 2$,

$$\text{Index}(\phi) + \text{Nullity}(\phi) \leq n(1 + 1/B)^B \{1 + 4B^2\},$$

where $B := {}^N R^{\phi}/\delta$.

The proof is omitted.

CHAPTER II. STABILITY OF THE IDENTITY MAP

4. A Kähler version of the Lichnerowicz-Obata theorem. In this chapter, we deal with the Jacobi operator of the identity map. Let (M, g) be a closed Riemannian manifold of dimension m . The identity map $\text{id}_M; (M, g) \rightarrow (M, g)$ of (M, g) is harmonic (cf. [E.S]), and the Riemannian manifold (M, g) is *stable* (cf. [Na]) if the identity map id_M is stable. The corresponding Jacobi operator $J := J_{\text{id}_M}$ is a differential operator acting on the space $\Gamma(TM)$ of all vector fields on M given by

$$(4.1) \quad JV = - \sum_{i=1}^m (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) - \rho(V), \quad V \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection of (M, g) , $\rho(V) := \sum_{i=1}^m R(e_i, V)e_i$, and

$$\rho(U, V) := g(\rho(U), V) = \sum_{i=1}^m g(R(e_i, U)e_i, V)$$

is the Ricci tensor (cf. [Ma, Sm]). Under the identification of TM with T^*M with respect to the metric g , the Hodge Laplacian $\Delta = d\delta + \delta d$ on $\Gamma(T^*M)$ induces a differential operator, denoted by the same letter and called also as the Hodge Laplacian, on $\Gamma(TM)$, where δ is the codifferential operator of d with respect to the metric g on M . Then the Weitzenböck formula for the Hodge operator Δ tells us that

$$(4.2) \quad \Delta V = - \sum_{i=1}^m (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) + \rho(V), \quad V \in \Gamma(TM),$$

so that

$$(4.3) \quad J = \Delta - 2\rho.$$

Then we have immediately

LEMMA 4.1. *Let $\lambda_1^1(M)$ (resp. $\lambda_1(M)$) be the first (resp. first nonzero) eigenvalue of the Hodge Laplacian (resp. the Laplace-Beltrami operator Δ_M) on 1-forms (resp. smooth functions) on M . Then*

(i) (M, g) is stable $\Rightarrow 2 \inf \text{Ric}_M \leq \lambda_1^1(M) \leq \lambda_1(M)$,

(ii) $\lambda_1^1(M) \geq 2 \sup \text{Ric}_M \Rightarrow (M, g)$ is stable,

where $\inf \text{Ric}_M$ (resp. $\sup \text{Ric}_M$) is the infimum (resp. supremum) of the Ricci curvature of (M, g) over M , $\inf \text{Ric}_M := \inf\{\rho(u, u); u \in TM, g(u, u) = 1\}$, and $\sup \text{Ric}_M := \sup\{\rho(u, u); u \in TM, g(u, u) = 1\}$.

PROOF. By (4.3), the stability of (M, g) implies that

$$\begin{aligned} 0 &\leq \int_M g(JV, V) * 1 = \int_M g(\Delta V, V) * 1 - 2 \int_M g(\rho(V), V) * 1 \\ &\leq \int_M g(\Delta V, V) * 1 - 2(\inf \text{Ric}_M) \int_M g(V, V) * 1, \end{aligned}$$

which gives the first inequality of (i). Taking V as the gradient of the eigenfunction of Δ_M with the eigenvalue $\lambda_1(M)$, we get the second inequality of (i). Statement (ii) is obvious from (4.3). Q.E.D.

From Lemma 4.1, we obtain

THEOREM 4.2 (M. OBATA). *Let (M, g) be a closed Kähler manifold whose Ricci curvature Ric_M is bounded below by a positive constant $\alpha > 0$. Then the first nonzero eigenvalue $\lambda_1(M)$ of Δ_M on $\mathcal{C}^\infty(M)$ satisfies $\lambda_1(M) \geq 2\alpha$. When the equality holds, the Lie algebra $\mathfrak{a}$ of the group of holomorphic transformations of M is nonzero.*

PROOF. Since every closed Kähler manifold (M, g) is stable (cf. [Sm, Na]), Lemma 4.1(i) gives the inequality $\lambda_1(M) \geq 2\alpha$. Assume that the equality $\lambda_1(M) = 2\alpha$ holds. We take V as the gradient of the eigenfunction of Δ_M with the eigenvalue 2α . Then $\Delta V = 2\alpha V$. By (4.3), we have

$$\begin{aligned} 2\alpha \int_M g(V, V) * 1 &= \int_M g(\Delta V, V) * 1 \\ &= \int_M g(JV, V) * 1 + 2 \int_M g(\rho(V), V) * 1 \\ &\geq 2\alpha \int_M g(V, V) * 1, \end{aligned}$$

since (M, g) is stable and $\text{Ric}_M \geq \alpha$. Hence we have $\int_M g(JV, V) * 1 = 0$ and $\int_M g(\rho(V), V) * 1 = \alpha \int_M g(V, V) * 1$. The former implies $JV = 0$, and then V belongs to $\mathfrak{a}$ due to a theorem of Lichnérowicz (cf. [L]) since (M, g) is a closed Kähler manifold. Q.E.D.

REMARK 1. In [Ob], the above theorem was stated for a closed Einstein Kähler manifold (M, g) . In this case, i.e., $\rho = \alpha g$, the equality $\lambda_1(M) = 2\alpha$ holds if and only if $\alpha \neq \{0\}$. The author does not know whether or not the equality holds if $\alpha \neq \{0\}$ without the assumption that (M, g) is Einstein.

REMARK 2. A theorem of Lichnérowicz-Obata tells us that for a closed Riemannian manifold (M, g) , if $\text{Ric}_M \geq \alpha = (n-1)\delta > 0$, then $\lambda_1(M) \geq n\delta = n\alpha/(n-1)$. Note that $n/(n-1) \leq 2$ and $n/(n-1) = 2 \Leftrightarrow n = 2$.

5. Some examples. In this section, we give three examples illustrating stability or instability of closed Riemannian manifolds.

5.1. By (4.1) and Corollary 2.2, we know (cf. [Sm]) that if Ricci curvature Ric_M of a closed Riemannian manifold (M, g) is nonpositive, then $\text{Index}(\text{id}_M) = 0$ and $\text{Index}(\text{id}_M) + \text{Nullity}(\text{id}_M) \leq m = \dim M$. Imitating the proof of Proposition 5.6 in [B.G, p. 30], noting only the difference of the constant terms of (4.1) and (4.2), we have

PROPOSITION 5.1. *There exists a positive constant $\epsilon_m > 0$ depending only on m such that for every closed Riemannian manifold (M, g) of dimension m with $\text{Ric}_M \leq \epsilon_m$, the index and nullity of the identity map of M satisfy $\text{Index}(\text{id}_M) + \text{Nullity}(\text{id}_M) \leq m$.*

However one cannot expect to find a positive constant $\epsilon_m > 0$ such that for every closed Riemannian manifold (M, g) of dimension m the assumption $\text{Ric}_M \leq \epsilon_m$ implies the stability of (M, g) , i.e., $\text{Index}(\text{id}_M) = 0$. In fact, we have the following example.

EXAMPLE 5.2. Let $\mathbf{T}^m = \mathbf{R}^m/\mathbf{Z}^m$ be the m -dimensional torus with the canonical coordinates $(x_1, \dots, x_m)$. Let $f(x_1)$ be a positive valued smooth function on $\mathbf{R}/\mathbf{Z} = S^1$. Consider the Riemannian metric g_f on $\mathbf{T}^m$ defined by

$$g_f := dx_1^2 + f(x_1)^2(dx_2^2 + \dots + dx_m^2).$$

LEMMA 5.3. *The vector field $X_1 = f(x_1)\partial/\partial x_1$ on $\mathbf{T}^m$ is a conformal vector field, i.e., the Lie derivative $L_{X_1}g_f$ of g_f by X_1 satisfies $L_{X_1}g_f = (2/n) \operatorname{div}(X_1)g_f$, and $X_i = \partial/\partial x_i$, $i = 2, \dots, m$, are Killing, i.e., $L_{X_i}g_f = 0$.*

Proof follows from a straightforward computation.

Since for a vector field V on a closed Riemannian manifold (M, g) ,

$$\int_M g(JV, V) * 1 = \int_M \left\{ \frac{1}{2} |L_V g|^2 - \operatorname{div}(V)^2 \right\} * 1,$$

where $|L_V g|$ is the norm of $L_V g$ induced by g and $\operatorname{div}(V)$ is the divergence of V (cf. [Y.B]), we have

$$\int_{\mathbf{T}^m} g(JX_1, X_1) * 1 = \left(\frac{2}{m} - 1 \right) \int_{\mathbf{T}^m} \operatorname{div}(X_1)^2 * 1.$$

Since $\operatorname{div}(X_1) = mf'(x_1)$ where $f'(x_1)$ is the derivative of $f(x_1)$, we have

PROPOSITION 5.4. *Let $\mathbf{T}^m = \mathbf{R}^m/\mathbf{Z}^m$ be the m -dimensional torus with the canonical coordinates $(x_1, \dots, x_m)$. For a positive valued smooth function $f(x_1)$ on $S^1 = \mathbf{R}/\mathbf{Z}$, consider the Riemannian metric g_f on $\mathbf{T}^m$ defined by*

$$g_f = dx_1^2 + f(x_1)^2(dx_2^2 + \dots + dx_m^2).$$

Then, in case $m \geq 3$, the Riemannian manifold $(\mathbf{T}^m, g_f)$ is stable if and only if the function $f(x_1)$ is constant.

On the other hand the sectional curvature K of the Riemannian manifold $(\mathbf{T}^m, g_f)$ is given (cf. [B.O]) as follows:

For each plane π in the tangent space $T_{(x_1, \dots, x_m)}\mathbf{T}^m$, let $\{x\partial/\partial x_1 + v, y\partial/\partial x_1 + w\}$ be an orthonormal basis of π , where $x, y \in \mathbf{R}$, and $v, w \in T_{(x_2, \dots, x_m)}\mathbf{T}^{m-1}$. Then the sectional curvature $K(\pi)$ is

$$\begin{aligned} K(\pi) = & -\frac{f''(x_1)}{f(x_1)} \left\{ x^2 g_f(w, w) - 2xy g_f(w, v) + y^2 g_f(v, v) \right\} \\ & - \frac{f'(x_1)^2}{f(x_1)^2} \left\{ g_f(v, v) g_f(w, w) - g_f(v, w)^2 \right\}. \end{aligned}$$

Thus the sectional curvature K of $(\mathbf{T}^m, g_f)$ satisfies

$$|K| \leq |f''|/f + f'^2/f^2.$$

For example, we can take a smooth function $f_\epsilon(x_1)$ on $S^1 = \mathbf{R}/\mathbf{Z}$ as $f_\epsilon(x_1) := 1 + \epsilon \sin(2\pi x_1)$, where ϵ is a small positive constant. Then due to Proposition 5.4, the Riemannian manifold $(\mathbf{T}^m, g_{f_\epsilon})$, $m \geq 3$, is unstable, but its sectional curvature K_ϵ satisfies

$$|K_\epsilon| \leq 4\pi^2 \left\{ \epsilon/(1 - \epsilon) + \epsilon^2/(1 - \epsilon)^2 \right\},$$

which goes to zero as $\varepsilon \rightarrow 0$. Therefore we cannot find a constant $\varepsilon_m > 0$ such that for every closed Riemannian manifold (M, g) of dimension m , the assumption $\text{Ric}_M \leq \varepsilon_m$ implies the stability of (M, g) .

5.2. The next example is the odd dimensional unit sphere S^{2n+1} , $n \geq 1$. Let $\phi; (S^{2n+1}, g) \rightarrow (\mathbb{C}P^n, h)$ be the Hopf fibration. Here g is the standard metric on S^{2n+1} of constant curvature one and h is the Fubini-Study metric on $\mathbb{C}P^n$ of constant holomorphic sectional curvature 4. Let ξ be the Killing vector field on (S^{2n+1}, g) such that $g(\xi, \xi) = 1$ and ξ is tangent to the fiber $\phi^{-1}(\phi(x))$ at each point x in S^{2n+1} . Let η be the 1-form dual to ξ . Then the projection $\phi; (S^{2n+1}, g) \rightarrow (\mathbb{C}P^n, h)$ is a Riemannian submersion with totally geodesic fibers (cf. §6) and $g = \phi^*h + \eta \otimes \eta$. Let us consider the canonical variation g_t , $0 < t < \infty$, of the metric g defined by

$$(5.1) \quad g_t := \phi^*h + t^2\eta \otimes \eta = g + (t^2 - 1)\eta \otimes \eta.$$

Now we investigate the stability of (S^{2n+1}, g_t) making use of Lemma 4.1.

To estimate the first eigenvalue $\lambda_1^1(g_t)$ of the Hodge Laplacian, put $m := 2n + 1$. Note that $g_t = s\{s^{-1}g + s^{-1}(s^m - 1)\eta \otimes \eta\}$, where $s := t^{2/m}$. In his paper [T1, Proposition 2.8], S. Tanno showed that the first eigenvalue $\lambda_1^1(g_t)$ of the Hodge Laplacian on 1-forms is estimated by

$$\lambda_1^1(g_t) \leq \text{Min}\{s^{-1} \cdot 2(m-1)s^{m+1}, s^{-1}(ms - s(1 - s^{-m}))\},$$

that is,

$$(5.2) \quad \lambda_1^1(g_t) \leq \text{Min}\{4nt^2, 2n + t^{-2}\}.$$

To study the Ricci curvature of (S^{2n+1}, g_t) , we recall some work of G. R. Jensen [J]. Set

$$K := SU(n+1),$$

$$H := S(U(n) \times U(1)) = \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & A \end{pmatrix} \in SU(n+1); \varepsilon \in U(1), A \in U(n) \right\},$$

$$H_1 := \left\{ \begin{pmatrix} \varepsilon & 0 \\ 0 & \gamma I_n \end{pmatrix}; \varepsilon \in U(1), \gamma = \varepsilon^{-1/n} \right\},$$

$$H_2 := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}; A \in SU(n) \right\},$$

where I_n is the unit matrix of order n . Then the natural projection gives the Hopf fibration $\phi; S^{2n+1} = K/H_2 \rightarrow \mathbb{C}P^n = K/H$. Let $\mathfrak{k}$ (resp. $\mathfrak{h}$, $\mathfrak{h}_1$, $\mathfrak{h}_2$) be the Lie algebra of K (resp. H , H_1 , H_2). Let F be the Killing form of $\mathfrak{k}$ and $\mathfrak{m}$, the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}$ with respect to F . Then we have the orthogonal decomposition of $\mathfrak{k}$: $\mathfrak{k} = \mathfrak{h}_2 \oplus \mathfrak{h}_1 \oplus \mathfrak{m}$. The metrics g_t in (5.1) are K -invariant on K/H_2 and come from the $\text{Ad}(H_2)$ -invariant inner product $\langle \cdot, \cdot \rangle_t$ on $\mathfrak{h}_1 \oplus \mathfrak{m}$ such that

$$\langle X_1 + X_2, Y_1 + Y_2 \rangle_t = (4(n+1))^{-1} \left\{ \frac{2n}{n+1} t^2 b(X_1, Y_1) + b(X_2, Y_2) \right\},$$

for $X_1, Y_1 \in \mathfrak{h}_1$, $X_2, Y_2 \in \mathfrak{m}$, where the inner product b on $\mathfrak{k}$ is given by $b = -F$. In fact, it is known that the restriction of b to $\mathfrak{m}$ coincides with $4(n+1)\pi^*h$, and $b(X, X) = 2(n+1)^2/n$ for

$$X := \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 0 & -n^{-1}I_n \end{pmatrix},$$

and ξ_o is the tangent vector at

$$o := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in S^{2n+1}$$

of the curve $\theta \mapsto \exp(\theta X) \cdot o$.

We denote by $S_{\tilde{g}}$ the Ricci tensor of the metric $\tilde{g}$ on K/H_2 corresponding to the inner product $4(n+1)\langle \cdot, \cdot \rangle_t$ on $\mathfrak{m}$. Then $S_{\tilde{g}}$ is a K -invariant tensor field on K/H_2 which is completely determined by the bilinear form on $\mathfrak{h}_1 \oplus \mathfrak{m}$, denoted by the same letter $S_{\tilde{g}}$. Note that the numbers k , c , r , and $\dim \mathfrak{m}$ in [J] are given in this case by $k = 1/2$, $c = 0$, $r = \dim \mathfrak{h}_1 = 1$, and $\dim \mathfrak{m} = 2n$. Thus by Proposition 11 in [J], the bilinear form $S_{\tilde{g}}$ is given by

$$\begin{aligned} S_{\tilde{g}}(X_1 + X_2, Y_1 + Y_2) &= \frac{1}{4} \left(\frac{2n}{n+1} \right) t^2 \cdot 4(n+1) \langle X_1, Y_1 \rangle_t \\ &\quad + \left(\frac{1}{2} - \frac{1}{4n} \left(\frac{2n}{n+1} \right) t^2 \right) \cdot 4(n+1) \langle X_2, Y_2 \rangle_t, \end{aligned}$$

$X_1, Y_1 \in \mathfrak{h}_1$, $X_2, Y_2 \in \mathfrak{m}$. Since

$$\begin{aligned} \text{Inf Ric}_{\tilde{g}} &= \text{Min} \left\{ \frac{1}{2} - \frac{t^2}{2(n+1)}, \frac{n}{2(n+1)} t^2 \right\}, \\ \text{Sup Ric}_{\tilde{g}} &= \text{Max} \left\{ \frac{1}{2} - \frac{t^2}{2(n+1)}, \frac{n}{2(n+1)} t^2 \right\}, \end{aligned}$$

it follows that

$$\begin{aligned} (5.3) \quad \text{Inf Ric}_{g_t} &= \text{Min} \{ 2(n+1) - 2t^2, 2nt^2 \}, \\ \text{Sup Ric}_{g_t} &= \text{Max} \{ 2(n+1) - 2t^2, 2nt^2 \}. \end{aligned}$$

To interpret (5.2) and (5.3), see Figure 5.1 in which $T = t^2$. Therefore we have

PROPOSITION 5.5. *Let g_t be the canonical variation (5.1) of the standard metric g of constant curvature one on S^{2n+1} with $g_t = g + (t^2 - 1)\eta \otimes \eta$. Then for every t^2 in the open interval (α, β) , the Riemannian manifold (S^{2n+1}, g_t) is unstable. Here $\alpha := (n + \sqrt{n^2 + 4n})/4n$ (resp. $\beta := (n + 2 + \sqrt{n^2 + 4n})/4$) is a root of the equation $4nT = 2n + T^{-1}$ (resp. $4(n+1) - 4T = 2n + T^{-1}$).*

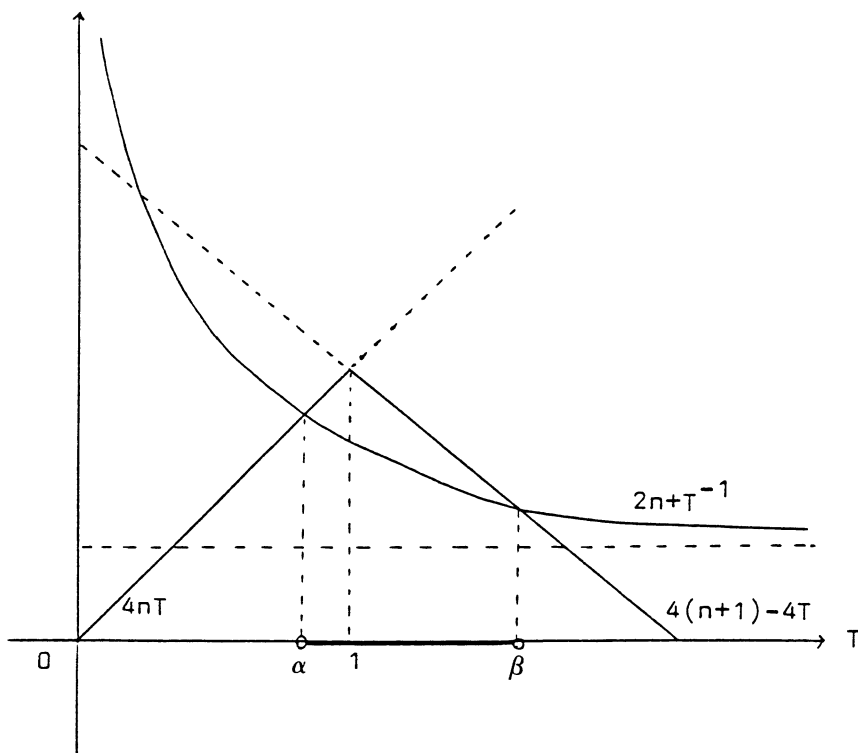


FIGURE 5.1. The graphs of the functions $4nT$, $2n + T^{-1}$, and $4(n+1) - 4T$

5.3. The third example is a spherical space form. Here we state the following:

PROPOSITION 5.6. *Every spherical space form $(S^n/G, g)$, where $G \neq \{\text{id}\}$ is a finite group acting fixed point freely on S^n , is stable. Here the metric g is the Riemannian metric on the quotient space S^n/G induced by the standard metric can of constant curvature one on S^n .*

In fact, this follows immediately from Proposition 2.1 in [Sm]. Since $(S^n/G, g)$ is Einstein, i.e., the Ricci tensor ρ of g satisfies $\rho = (n-1)g$, the manifold $(S^n/G, g)$ is stable if and only if the first nonzero eigenvalue $\lambda_1(S^n/G, g)$ of the Laplace-Beltrami operator Δ_M on $\mathcal{C}^\infty(S^n/G)$ is bigger than or equal to $2(n-1)$. The eigenvalues of Δ_M of (S^n, can) are given by $k(k+n-1)$, $k = 0, 1, 2, \dots$, and $k(k+n-1) > 2(n-1)$ if $k \geq 2$. Moreover the eigenfunctions of the first nonzero eigenvalue n with $k = 1$ of (S^n, can) are given by $F \cdot \text{id}_{S^n}$, where F is a linear map of $\mathbf{R}^{n+1}$ into $\mathbf{R}$ and id_{S^n} is the natural inclusion of S^n into $\mathbf{R}^{n+1}$. Therefore we only have to show that every linear G -invariant function F on $\mathbf{R}^{n+1}$ must be zero. But this follows immediately from the assumption that G acts fixed point freely on S^n . Certainly, $F(x) = \langle x, y \rangle$, $x \in \mathbf{R}^{n+1}$, for some y in $\mathbf{R}^{n+1}$. The G -invariance of F implies that $\gamma \cdot y = y$ for all $\gamma \in G$. Unless F vanishes, the point $y/|y| \in S^n$ must be a fixed point of G .

Since every compact Riemannian manifold of positive constant curvature is as in Proposition 5.6 (cf. [W, Lemma 5.11, p. 154]) and every compact Riemannian manifold of constant zero or negative curvature is stable (cf. [Sm]), we have

COROLLARY 5.7. *Every compact Riemannian manifold of constant curvature is stable except for the standard unit sphere (S^n , can).*

REMARK. A similar stability theorem for Yang-Mills fields was stated in [B.L, p. 223].

CHAPTER III. RIEMANNIAN SUBMERSIONS WITH TOTALLY GEODESIC FIBERS

6. The vertical Jacobi operator.

6.1. Following [O.N] or [B.B], let us recall the definition of a Riemannian submersion. It is known (cf. [E.L, p. 127]) that the projection of a Riemannian submersion is harmonic if and only if each fiber of the submersion is a minimal submanifold. In particular, the projection of the Riemannian submersion with totally geodesic fibers is harmonic. The Riemannian submersions are the next simple examples after Riemannian products, but would be rich objects to study. In this chapter, we study Jacobi operators of projections of Riemannian submersions with totally geodesic fibers by analogy with the theory of Laplace-Beltrami operators (cf. [B.B]).

DEFINITION 6.1. Let (M, g) and (N, h) be two closed Riemannian manifolds of dimension m and n respectively. A map $\phi; (M, g) \rightarrow (N, h)$ is a *Riemannian submersion* (cf. [O.N, B.B]) if for each point p in M , the tangent space $T_p M$ of M at p has an orthogonal decomposition $T_p M = H_p \oplus V_p$ with respect to g_p such that

(i) the subspace V_p is the kernel of the differential ϕ_{*p} of ϕ at p , which is called the *vertical space*, and

(ii) the restriction of ϕ_{*p} to the subspace H_p , called the *horizontal space*, is an isometry of (H_p, g_p) onto $(T_{\phi(p)} N, h_{\phi(p)})$. A vector field X on $U \subset M$ is called *basic* if it is the horizontal lift of a vector field X' on $\phi(U) \subset N$. In this chapter, we further assume that each fiber $F_p := \phi^{-1}(\phi(p))$ through p admitting the Riemannian metric induced from g is totally geodesic in (M, g) .

6.2. To define the vertical Jacobi operator, we take an orthonormal local frame field $\{e_i\}_{i=1}^m$ on M such that

(i) for $1 \leq i \leq n$, each e_i is basic, the horizontal lift of e'_i , with $\{e'_i\}_{i=1}^n$ an orthonormal local frame field on N , and

(ii) for $n+1 \leq i \leq m$, each e_i is vertical.

Then it is known (cf. [O.N or B.B]) that $\nabla_{e_i} e_i$, $1 \leq i \leq n$, is basic, the horizontal lift of ${}^N \nabla_{e'_i} e'_i$, while $\nabla_{e_i} e_i$, $n+1 \leq i \leq m$, is vertical since all the fibers are totally geodesic. In the following we retain the notation of §1.

DEFINITION 6.2. Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers and J_ϕ , the Jacobi operator acting on $\Gamma(\phi^{-1}TN)$. We define the *vertical Jacobi operator* acting on $\Gamma(\phi^{-1}TN)$ by

$$J_\phi^v := - \sum_{i=n+1}^m \left(\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i} \right),$$

and the *horizontal* Jacobi operator acting on $\Gamma(\phi^{-1}TN)$ by $J_\phi^H := J_\phi - J_\phi^v$. Then it is easy to see that the definitions of J_ϕ^v and J_ϕ^H do not depend on the above choice of the orthonormal local frame field $\{e_i\}_{i=1}^m$ on M (cf. Remark below). These definitions give analogues of the vertical and horizontal Laplacians Δ_v and Δ_H acting on $\mathcal{C}^\infty(M)$ defined in [B.B] by $\Delta_v := \sum_{i=n+1}^m (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})$ and $\Delta_H := \Delta_M - \Delta_v$, where $\Delta_M := \sum_{i=1}^m (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})$ is the Laplacian-Beltrami operator of (M, g) . Note that Δ_v , Δ_H , and Δ_M commute (cf. [B.B, Theorem 1.5]).

Each section W in $\Gamma(\phi^{-1}TN)$ can be expressed locally as

$$(6.1) \quad W = \sum_{i=1}^n f_i \widetilde{e'_i},$$

where the f_i , $1 \leq i \leq n$, are locally defined smooth functions on M and the $\widetilde{e'_i}$, $1 \leq i \leq n$, are local sections of $\phi^{-1}TN$ defined by $\widetilde{e'_{ix}} := e'_{i\phi(x)}$, $x \in M$. Then by definition of $\tilde{\nabla}$ and $\phi_* e_i = 0$, $n+1 \leq i \leq m$, we have

$$(6.2) \quad \tilde{\nabla}_{e_i} W = \sum_{j=1}^n \left\{ (e_i f_j) \widetilde{e'_j} + f_j \tilde{\nabla}_{e_i} \widetilde{e'_j} \right\}, \quad 1 \leq i \leq m,$$

$$(6.2') \quad \tilde{\nabla}_{e_i} W = \sum_{j=1}^n (e'_i f_j) \widetilde{e'_j}, \quad n+1 \leq i \leq m.$$

In particular,

$$(6.3) \quad J_\phi^v W = - \sum_{j=1}^n (\Delta_v f_j) \widetilde{e'_j}.$$

REMARK. The intrinsic meaning of the vertical Jacobi operator is described as follows. For each fiber $F_p = \phi^{-1}(\phi(p))$ through $p \in M$, the composition $\phi \circ i_p$; $F_p \rightarrow N$ of the inclusion i_p of F_p into M and the projection ϕ is constant, so harmonic. The associate Jacobi operator $J_{\phi \circ i_p}$ acting on $\Gamma((\phi \circ i_p)^{-1}TN)$ is well defined. Then $\Gamma((\phi \circ i_p)^{-1}TN)$ consists of all the restrictions, $W|_{F_p}$, to F_p of elements W in $\Gamma(\phi^{-1}TN)$ and

$$(J_\phi^v W)(p) = J_{\phi \circ i_p}(W|_{F_p})(p), \quad W \in \Gamma(\phi^{-1}TN).$$

6.3. We now describe some fundamental properties of J_ϕ^v and J_ϕ^H . Note that, by the definitions of $\tilde{\nabla}$ and $\widetilde{W'}$,

$$(6.4) \quad \tilde{\nabla}_{e_i} \widetilde{W'} = \widetilde{N_{\nabla_{\phi_* e_i}} W'} = \begin{cases} \widetilde{N_{\nabla_{e'_i}} W'}, & 1 \leq i \leq n, \\ 0, & n+1 \leq i \leq m, \end{cases}$$

for $W' \in \Gamma(TN)$. Then we have

$$(6.5) \quad J_\phi^v(\widetilde{W'}) = 0, \quad \text{and} \quad J_\phi^H(\widetilde{W'}) = \widetilde{J_{\text{id}_N}(W')},$$

for $W' \in \Gamma(TN)$, by (6.4) and definition of J_ϕ^v and J_ϕ^H . Therefore we obtain

PROPOSITION 6.3. *Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers. Then*

$$\text{Index}(\phi) \geq \text{Index}(\text{id}_N), \quad \text{Nullity}(\phi) \geq \text{Nullity}(\text{id}_N),$$

and $\lambda_1(J_\phi) \leq \lambda_1(J_{\text{id}_N})$. In particular, if the base manifold (N, h) is unstable, then the submersion ϕ is unstable.

Proof follows immediately from definitions and (6.5).

REMARK. The referee pointed out that Proposition 6.3 was obtained independently by M. J. Ferreira in his Ph.D. Thesis. Moreover, throughout §§6.1–6.3 the assumption that the fibers of the Riemannian submersions must be totally geodesic can be replaced by the assumption that the Riemannian submersions are harmonic.

PROPOSITION 6.4. (i) *Let $F = F_p$ be the fiber through $p \in M$ of a Riemannian submersion $\phi; (M, g) \rightarrow (N, h)$ with totally geodesic fibers. For each $W \in \Gamma(\phi^{-1}TN)$, we have*

$$\int_F h(J_\phi^v W, W) dv_F = \sum_{i=n+1}^m \int_F h(\tilde{\nabla}_{e_i} W, \tilde{\nabla}_{e_i} W) dv_F,$$

where dv_F is the volume element on F with respect to the metric g_F induced by the metric g on M .

(ii) *Moreover, for each $W \in \Gamma(\phi^{-1}TN)$, $J_\phi^v W = 0$ if and only if $W = \widetilde{W'}$ for some $W' \in \Gamma(TN)$.*

(iii) *Each eigenvalue of J_ϕ^v is nonnegative.*

PROOF. (i) For each $W \in \Gamma(\phi^{-1}TN)$, we have

$$\begin{aligned} h(J_\phi^v W, W) &= - \sum_{i=n+1}^m e_i \cdot h(\tilde{\nabla}_{e_i} W, W) + \sum_{i=n+1}^m h(\tilde{\nabla}_{e_i} W, \tilde{\nabla}_{e_i} W) \\ &\quad + \sum_{i=n+1}^m h(\tilde{\nabla}_{\nabla_{e_i} e_i} W, W). \end{aligned}$$

Here there exists an element X in $\Gamma(TF)$ such that $g_F(X, Y) = h(\tilde{\nabla}_Y W, W)$ for each $Y \in \Gamma(TF)$. Then since $\nabla_{e_i} e_i$, $n+1 \leq i \leq m$, are vertical

$$\begin{aligned} &\sum_{i=n+1}^m \left\{ e_i \cdot h(\tilde{\nabla}_{e_i} W, W) - h(\tilde{\nabla}_{\nabla_{e_i} e_i} W, W) \right\} \\ &= \sum_{i=n+1}^m \left\{ e_i \cdot g_F(X, e_i) - g_F(\nabla_{e_i} e_i, X) \right\} \end{aligned}$$

is the gradient of X on (F, g_F) . Therefore we have (i).

(ii) By (6.5), we only have to prove that if $J_\phi^v W = 0$, then $W = \widetilde{W'}$ for some $W' \in \Gamma(TN)$. Assume that $J_\phi^v W = 0$. Then by (i) we have $\tilde{\nabla}_{e_i} W = 0$, $n+1 \leq i \leq m$. We choose a local coordinate system $(x_U^1, \dots, x_U^n)$ on a neighborhood U in N . Then W can be expressed locally as

$$W_x = \sum_{j=1}^n f_{U,j}(x) \left(\frac{\partial}{\partial x_U^j} \right)_{\phi(x)}, \quad x \in \phi^{-1}(U),$$

where $f_{U,j} \in \mathcal{C}^\infty(\phi^{-1}(U))$. Since $W \in \Gamma(\phi^{-1}TN)$, it satisfies

$$(6.6) \quad f_{U,i} = \sum_{j=1}^n f_{V,j} \frac{\partial x_U^i}{\partial x_V^j}$$

on $\phi^{-1}(U) \cap \phi^{-1}(V)$ for another coordinate system $(x_V^1, \dots, x_V^n)$ on V . By (6.2') $0 = \tilde{\nabla}_{e_i} W = \sum_{j=1}^n (e_i f_{U,j}) (\partial/\partial x_V^j)$. Therefore $e_i f_{U,j} = 0$, $n+1 \leq i \leq m$, that is, the $f_{U,j}$ are constant along each fiber, which implies that $f_{U,j} = f'_{U,j} \circ \phi$ for some $f'_{U,j} \in \mathcal{C}^\infty(U)$. By (6.6), $f'_{U,j}$ satisfies

$$f'_{U,i} = \sum_{j=1}^n f'_{V,j} (\partial x_U^i / \partial x_V^j) \quad \text{on } U \cap V.$$

Therefore $\{\sum_{j=1}^n f'_{U,j} \partial/\partial x_U^j\}$ defines a section W' in $\Gamma(TN)$ such that $W = \widetilde{W'}$.

Finally, (iii) follows immediately from (i). Q.E.D.

6.4. This section is devoted to the following result.

THEOREM 6.5. *Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers. Then the operators J_ϕ^v , J_ϕ^H and J_ϕ commute.*

PROOF. We only have to prove that $J_\phi^v J_\phi^H = J_\phi^H J_\phi^v$. For each $W \in \Gamma(\phi^{-1}TN)$, we have

$$\begin{aligned} (6.7) \quad J_\phi^H J_\phi^v W &= - \sum_{j,k=1}^n \left\{ (\tilde{\nabla}_{e_k} \tilde{\nabla}_{e_k} - \tilde{\nabla}_{\nabla_{e_k} e_k}) ((\Delta_v f_j) \widetilde{e'_j}) + (\Delta_v f_j) {}^N R(e'_k, e'_j) e'_k \right\} \\ &= - \sum_{j,k=1}^n \left\{ e_k^2 (\Delta_v f_j) \widetilde{e'_j} + 2e_k (\Delta_v f_j) \tilde{\nabla}_{e_k} e'_j + (\Delta_v f_j) \tilde{\nabla}_{e_k} \tilde{\nabla}_{e_k} \widetilde{e'_j} \right. \\ &\quad \left. - (\nabla_{e_k} e_k) (\Delta_v f_j) \widetilde{e'_j} - (\Delta_v f_u) \tilde{\nabla}_{\nabla_{e_k} e_k} \widetilde{e'_j} \right\} \\ &\quad - \sum_{j,k=1}^n (\Delta_v f_j) {}^N R(e'_k, e'_j) e'_k, \end{aligned}$$

by definition of J_ϕ^H and (6.3). Since the e_k and $\nabla_{e_k} e_k$, $1 \leq k \leq n$, are basic, and Δ_v commutes with basic vector fields (cf. [B.B, Lemma 1.6]), the first term of the right-hand side of (6.7) becomes

$$\begin{aligned} &- \sum_{j,k=1}^n \left\{ \Delta_v (e_k^2 f_j) \widetilde{e'_j} + 2\Delta_v (e_k f_j) \tilde{\nabla}_{e_k} \widetilde{e'_j} + (\Delta_v f_j) \tilde{\nabla}_{e_k} \tilde{\nabla}_{e_k} \widetilde{e'_j} \right. \\ &\quad \left. - \Delta_v (\nabla_{e_k} e_k f_j) \widetilde{e'_j} - (\Delta_v f_j) \tilde{\nabla}_{\nabla_{e_k} e_k} \widetilde{e'_j} \right\} \\ &= - \sum_{j,k=1}^n J_\phi^v \left\{ (e_k^2 f_j) \widetilde{e'_j} + 2(e_k f_j) \tilde{\nabla}_{e_k} \widetilde{e'_j} + f_j \tilde{\nabla}_{e_k} \tilde{\nabla}_{e_k} \widetilde{e'_j} \right. \\ &\quad \left. - (\nabla_{e_k} e_k f_j) \widetilde{e'_j} - f_j \tilde{\nabla}_{\nabla_{e_k} e_k} \widetilde{e'_j} \right\}, \end{aligned}$$

by (6.3) and (6.4). Therefore we obtain

$$J_\phi^H J_\phi^v W = - \sum_{k=1}^n J_\phi^v \left\{ (\tilde{\nabla}_{e_k} \tilde{\nabla}_{e_k} - \tilde{\nabla}_{\nabla_{e_k} e_k}) W - {}^N R(e'_k, W) e'_k \right\} = J_\phi^v J_\phi^H W.$$

Q.E.D.

We have immediately

COROLLARY 6.6. *The Hilbert space of all L^2 sections of $\phi^{-1}TN$ with respect to the inner product $(V, W) := \int_M h(V, W) * 1$, for sections V, W , has a complete orthonormal basis consisting of the simultaneous eigensections of J_ϕ^v , J_ϕ^H , and J_ϕ .*

7. The canonical variation of a Riemannian submersion.

7.1. We continue the discussion in §6. Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers.

DEFINITION 7.1 (cf. [B.B, p. 191]). For each positive real number t , let g_t be the unique Riemannian metric on M such that

- (i) $g_t(u, v) = g(u, v)$ for $u, v \in H_p$, $p \in M$,
- (ii) the subspaces H_p and V_p are orthogonal to each other with respect to g_t at each point p in M , and
- (iii) $g_t(u, v) = t^2 g(u, v)$ for $u, v \in V_p$, $p \in M$.

Then $\phi; (M, g_t) \rightarrow (N, h)$ is a Riemannian submersion with totally geodesic fibers (cf. [B.B, Proposition 5.2]), which is called the *canonical variation*.

For each $t > 0$, $\{e_1, \dots, e_n, t^{-1}e_{n+1}, \dots, t^{-1}e_m\}$ is an orthonormal local frame field on (M, g_t) with e_i the horizontal lift of e'_i with respect to g_t for $1 \leq i \leq n$, and with $t^{-1}e_i$ vertical for $n+1 \leq i \leq m$. Then the vertical (resp. horizontal) Jacobi operator ${}^tJ_\phi^v$ (resp. ${}^tJ_\phi^H$) of the canonical variation $\phi; (M, g_t) \rightarrow (N, h)$ satisfies

$${}^tJ_\phi^v = t^{-2}J_\phi^v, \quad \text{and} \quad {}^tJ_\phi^H = J_\phi^H.$$

Therefore we have

PROPOSITION 7.2. *The following formula holds:*

$${}^tJ_\phi = t^{-2}J_\phi^v + J_\phi^H = t^{-2}J_\phi + (1 - t^{-2})J_\phi^H.$$

REMARK. This is the analogue of Proposition 5.3 in [B.B].

7.2. Due to Corollary 6.6 and Proposition 7.2, each eigenvalue of ${}^tJ_\phi$ can be written as

$$(7.1) \quad \lambda + t^{-2}\mu,$$

where λ is an eigenvalue of J_ϕ^H and $\mu \geq 0$ is an eigenvalue of J_ϕ^v . Then the following two cases occur:

- (i) $\mu > 0$, or
- (ii) $\mu = 0$.

In case (i), $\lambda + t^{-2}\mu$ goes to infinity when $t \rightarrow 0$. In case (ii), $\lambda + t^{-2}\mu = \lambda$ which does not depend on t . Since the number of the eigenvalues of J_ϕ smaller than a given number is finite, there exists a small positive number ε such that for each $0 < t < \varepsilon$, the first eigenvalue $\lambda_1({}^tJ_\phi)$ coincides with the smallest eigenvalue of ${}^tJ_\phi$ when the case (ii) occurs. Then we have

$$\begin{aligned} \lambda_1({}^tJ_\phi) &= \text{Min}\{\lambda; J_\phi W = \lambda W \text{ and } J_\phi^v W = 0 \text{ for some } 0 \neq W \in \Gamma(\phi^{-1}TN)\} \\ &= \lambda_1(J_{\text{id}_N}), \end{aligned}$$

because of Propositions 6.4(ii) and 6.3. Therefore we obtain

THEOREM 7.3. *Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers, and let g_t , $0 < t < \infty$, be the canonical variation (cf. Definition 7.1) of g with $g_1 = g$. Then there exists a number $\varepsilon > 0$ such that for each $0 < t < \varepsilon$,*

$$\lambda_1(J_\phi) = \lambda_1(J_{\text{id}_N}).$$

In particular, if (N, h) is stable, then the submersion $\phi; (M, g) \rightarrow (N, h)$ is stable for every $0 < t < \varepsilon$.

7.3. Typical examples of a Riemannian submersion with totally geodesic fibers are the homogeneous Riemannian submersions (cf. [B.B, §2]). Let G be a compact connected Lie group, and K, H closed subgroups of G . Let $\mathfrak{g}$ (resp. $\mathfrak{k}, \mathfrak{h}$) be the Lie algebra of G (resp. K, H). We choose subspaces $\mathfrak{h}_1$ (resp. $\mathfrak{p}$) of $\mathfrak{k}$ (resp. $\mathfrak{g}$) such that $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{h}_1$, with $\text{Ad}(H)\mathfrak{h}_1 = \mathfrak{h}_1$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with $\text{Ad}(K)\mathfrak{p} = \mathfrak{p}$. Put $\mathfrak{m} := \mathfrak{h}_1 \oplus \mathfrak{p}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, with $\text{Ad}(H)\mathfrak{m} = \mathfrak{m}$. Let $(\cdot, \cdot)_{\mathfrak{h}_1}$ (resp. $(\cdot, \cdot)_{\mathfrak{p}}$) be an $\text{Ad}(H)$ -invariant (resp. $\text{Ad}(K)$ -invariant) inner product on $\mathfrak{h}_1$ (resp. $\mathfrak{p}$). We define an $\text{Ad}(H)$ -invariant inner product $(\cdot, \cdot)_{\mathfrak{m}}$ on $\mathfrak{m}$ by

$$(X_1 + X_2, Y_1 + Y_2)_{\mathfrak{m}} := (X_1, Y_1)_{\mathfrak{h}_1} + (X_2, Y_2)_{\mathfrak{p}}, \quad X_1, Y_1 \in \mathfrak{h}_1, X_2, Y_2 \in \mathfrak{p}.$$

Then the inner product $(\cdot, \cdot)_{\mathfrak{h}_1}$ (resp. $(\cdot, \cdot)_{\mathfrak{p}}, (\cdot, \cdot)_{\mathfrak{m}}$) gives a K -invariant (resp. G -invariant) Riemannian metric k (resp. h, g) on K/H (resp. $G/K, G/H$). It is known (cf. [B.B]) that the projection $\phi; G/H \ni xH \mapsto xK \in G/K$ gives a Riemannian submersion of $(G/H, g)$ onto $(G/K, h)$ with totally geodesic fibers $(K/H, k)$.

In particular, these give the Hopf fibrations:

- (i) $\phi_1; S^{4n+3} = \text{Sp}(n+1)/\text{Sp}(n) \rightarrow \mathbf{HP}^n = \text{Sp}(n+1)/\text{Sp}(1) \times \text{Sp}(n)$,
- (ii) $\phi_2; S^{2n+1} = \text{SU}(n+1)/\text{SU}(n) \rightarrow \mathbf{CP}^n = \text{SU}(n+1)/\text{S}(\text{U}(1) \times \text{U}(n))$.

Note that $\text{Sp}(n+1)$ -invariant (resp. $\text{SU}(n+1)$ -invariant) metrics h on $\mathbf{HP}^n$ (resp. $\mathbf{CP}^n$) are unique up to a constant factor.

Since $(\mathbf{HP}^n, h)$ (resp. $(\mathbf{CP}^n, h)$) is unstable (resp. stable) (cf. [Sm, Na]), we have

PROPOSITION 7.4. (i) *For each $\text{Sp}(n+1)$ -invariant metric g on $S^{4n+3} = \text{Sp}(n+1)/\text{Sp}(n)$, the Riemannian submersion $\phi_1; (S^{4n+3}, g) \rightarrow (\mathbf{HP}^n, h)$ is unstable.*

(ii) *For each $\text{SU}(n+1)$ -invariant metric g on $S^{2n+1} = \text{SU}(n+1)/\text{SU}(n)$, there exists a number $\varepsilon > 0$ such that for each $0 < t < \varepsilon$, the canonical variation $\phi_2; (S^{2n+1}, g_t) \rightarrow (\mathbf{CP}^n, h)$ is stable.*

The proof follows from Proposition 6.3 and Theorem 7.3.

REMARK. Proposition 7.4 asserts that each odd dimensional unit sphere S^{2n+1} , $n \geq 1$, with the canonical variation g_t , $0 < t < \varepsilon$, admits a nonconstant stable harmonic map. By way of contrast, Y. L. Xin [X] showed that each nonconstant harmonic map from the standard unit sphere (S^m, can) , $m \geq 3$, of constant curvature into an arbitrary Riemannian manifold is unstable.

7.4. Next, let us study the case in which t goes to infinity. We retain the notation of §7.1. Let us recall that the *holonomy group* G of a fiber F of the submersion $\phi; (M, g) \rightarrow (N, h)$ with totally geodesic fibers is the group of all isometries of the fiber F induced by the horizontal transports along the horizontal lifts of loops in N based

at the projection of F . It is known [O.N, Theorem 5] that $G = \{\text{id}\}$ if and only if the submersion $\phi; (M, g) \rightarrow (N, h)$ is trivial, that is, there exist an isometry ι of (M, g) and a submanifold F of M such that M is the Riemannian product $F \times N$ and $\phi = \text{pr} \circ \iota$, where pr is the projection of $F \times N$ onto N .

THEOREM 7.5. *Let $\phi; (M, g) \rightarrow (N, h)$ be a Riemannian submersion with totally geodesic fibers. Assume that the holonomy group G of a fiber F of the submersion $\phi; (M, g) \rightarrow (N, h)$ does not act transitively on the fiber, and $\text{Index}(\text{id}_N) > 0$. Then the index of the canonical variation $\phi; (M, g_t) \rightarrow (N, h)$ goes to infinity when $t \rightarrow \infty$.*

PROOF. Let $\mathcal{C}_G^\infty(F)$ be the space of all functions f in $\mathcal{C}^\infty(F)$ invariant under the actions of G . Since each G -orbit has an open G -invariant tubular neighborhood in M (cf. [Br, Theorem 2.2, p. 306]), there exists a nonconstant function f in $\mathcal{C}_G^\infty(F)$. Then the dimension of $\mathcal{C}_G^\infty(F)$ is infinite. Each element f in $\mathcal{C}_G^\infty(F)$ can be extended to a function $\tilde{f}$ in the space $\mathcal{C}_v^\infty(M)$ of all elements in $\mathcal{C}^\infty(M)$ which are invariant under horizontal transport. Since parallel transport is an isometry, the vertical Laplacian Δ_v leaves $\mathcal{C}_v^\infty(M)$ invariant. Therefore there exist an infinite number of eigenvalues $0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_i \leq \dots$, of Δ_v counted with their multiplicities such that

$$(7.2) \quad -\Delta_v f_i = \mu_i f_i, \quad 0 \neq f_i \in \mathcal{C}_v^\infty(M), \quad i = 1, 2, \dots$$

Now suppose that $\text{Index}(\text{id}_N) > 0$, that is, there exists a nonzero element W' in $\Gamma(TN)$ such that $J_{\text{id}_N} W' = \lambda W'$ and $\lambda < 0$. By Proposition 7.2, (6.3), $f_i \in \mathcal{C}_v^\infty(M)$, (6.5) and (7.2), we have

$$\begin{aligned} J_\phi(f_i \widetilde{W'}) &= (t^{-2} J_\phi^v + J_\phi^H)(f_i \widetilde{W'}) \\ &= t^{-2} (-\Delta_v f_i) \widetilde{W'} + f_i J_\phi^H(\widetilde{W'}) = (t^{-2} \mu_i + \lambda)(f_i \widetilde{W'}). \end{aligned}$$

That is, J_ϕ has the eigenvalues $t^{-2} \mu_i + \lambda$, $i = 1, 2, \dots$. When t goes to infinity, the eigenvalues $t^{-2} \mu_i + \lambda$ tend to the eigenvalue λ . Since $\lambda < 0$, for each $i = 1, 2, \dots$, there exists a number $N > 0$ such that $t^{-2} \mu_i + \lambda < 0$ for $t \geq N$. Therefore we have the desired conclusion.

REMARK. Theorem 7.5 is a generalization of Corollary 3.3 in [Sm].

8. Homogeneous Riemannian submersions.

8.1. In this section, we express the Jacobi operator of homogeneous Riemannian submersions in terms of Lie algebras and calculate the spectrum of the Jacobi operator of the Hopf fibration. We retain the notation of §7.3.

Let G be a compact connected Lie group, with K and H closed subgroups of G . Let $\mathfrak{g}$ be the Lie algebra of G consisting of all left invariant vector fields on G . Let $\mathfrak{k}$ and $\mathfrak{h}$ be the subalgebras corresponding to K and H . Put $s := \dim G$, $m := \dim G/H$, and $n := \dim G/K$. We choose an $\text{Ad}(G)$ -invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$, with $\mathfrak{h}_1$ (resp. $\mathfrak{p}$), the orthogonal complement of $\mathfrak{h}$ (resp. $\mathfrak{k}$) in $\mathfrak{k}$ (resp. $\mathfrak{g}$). Then

$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{h}_1$ with $\text{Ad}(H)\mathfrak{h}_1 = \mathfrak{h}_1$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\text{Ad}(K)\mathfrak{p} = \mathfrak{p}$. Put $\mathfrak{m} := \mathfrak{h}_1 \oplus \mathfrak{p}$, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\text{Ad}(H)\mathfrak{m} = \mathfrak{m}$. In this section, we always assume the following:

ASSUMPTION (A). We take the inner products $(\cdot, \cdot)_{\mathfrak{h}_1}$, $(\cdot, \cdot)_{\mathfrak{p}}$, and $(\cdot, \cdot)_{\mathfrak{m}}$ as the restrictions to $\mathfrak{h}_1$, $\mathfrak{p}$, and $\mathfrak{m}$, respectively, of the above $\text{Ad}(G)$ -invariant inner product $(\cdot, \cdot)$ on $\mathfrak{g}$.

Now we consider the Riemannian submersion $\phi; G/H \rightarrow G/K$ admitting the Riemannian metric g (resp. h) on G/H (resp. G/K) corresponding to the inner product $(\cdot, \cdot)$ on $\mathfrak{m}$ (resp. $\mathfrak{p}$). Since the induced bundle $E := \phi^{-1}T(G/K)$ is identified with the associate bundle $G \times_H \mathfrak{p}$, which is the space of the equivalence classes of $(x, X) \in G \times \mathfrak{p}$ under the equivalence relation $(xh, \text{Ad}(h)X) \sim (x, X)$, for $h \in H$, we can identify the space $\Gamma(E)$ of its sections with the following space.

DEFINITION 8.1. Let $\mathcal{C}^\infty(G, \mathfrak{p})$ be the space of all smooth maps of G into $\mathfrak{p}$. We define the subspace $\mathcal{C}_H^\infty(G, \mathfrak{p})$ of $\mathcal{C}^\infty(G, \mathfrak{p})$ by

$$\mathcal{C}_H^\infty(G, \mathfrak{p}) := \{ f \in \mathcal{C}^\infty(G, \mathfrak{p}); f(xh) = \text{Ad}(h^{-1})f(x), x \in G, h \in H \}.$$

The identification Φ of $\Gamma(E)$ with $\mathcal{C}_H^\infty(G, \mathfrak{p})$, $\Phi; \mathcal{C}_H^\infty(G, \mathfrak{p}) \rightarrow \Gamma(E)$, is given by

$$(8.1) \quad \Phi(f)(xH) := \tau_{x*}f(x)_{\{K\}}, \quad x \in G.$$

Here $f(x)_{\{K\}}$ is the tangent vector of G/K at the origin $\{K\}$ corresponding to $f(x) \in \mathfrak{p}$, and τ_{x*} is the differential of the translation $\tau_x; G/K \ni yK \mapsto xyK \in G/K$. Then it turns out that Φ is an isomorphism of $\mathcal{C}_H^\infty(G, \mathfrak{p})$ onto $\Gamma(E)$. Under the G -actions on $\Gamma(E)$ or $\mathcal{C}_H^\infty(G, \mathfrak{p})$ defined by

$$\begin{aligned} (\tau_x V)_{yH} &:= \tau_{x*}V_{x^{-1}yH}, & x, y \in G, V \in \Gamma(E), \\ (\tau_x f)(y) &:= f(x^{-1}y), & x, y \in G, f \in \mathcal{C}_H^\infty(G, \mathfrak{p}), \end{aligned}$$

Φ is a G -isomorphism, that is,

$$(8.2) \quad \Phi \circ \tau_x f = \tau_{x*} \Phi(f), \quad x \in G, f \in \mathcal{C}_H^\infty(G, \mathfrak{p}).$$

Note that the Jacobi operator $J_\phi; \Gamma(E) \rightarrow \Gamma(E)$ is G -invariant, that is,

$$(8.3) \quad J_\phi(\tau_x V) = \tau_{x*}(J_\phi V), \quad V \in \Gamma(E).$$

Here we denote by τ_{x*} the differential of the translation τ_x on G/H or G/K by $x \in G$. Then we have $\tau_{x^{-1}*}\nabla_{e_i}e_i = \nabla_{\tau_{x^{-1}*}e_i}\tau_{x^{-1}*}e_i$, and $\tilde{\nabla}_{e_i}\tau_{x*}V = \tau_{x*}\tilde{\nabla}_{\tau_{x^{-1}*}e_i}V$, for $V \in \Gamma(E)$, $x \in G$, where $\{e_i\}_{i=1}^m$ is an orthonormal local frames field on $(G/H, g)$. Because of the expression (1.4) for J_ϕ , we have the G -invariance of J_ϕ .

Furthermore we identify $\mathcal{C}_H^\infty(G, \mathfrak{p})$ with the subspace $(\mathcal{C}^\infty(g) \otimes \mathfrak{p})_H$ of the tensor product $\mathcal{C}^\infty(G) \otimes \mathfrak{p}$.

DEFINITION 8.2. $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ is by definition the subspace of $\mathcal{C}^\infty(G) \otimes \mathfrak{p}$ consisting of all elements $\sum_{i=1}^l f_i \otimes X_i \in \mathcal{C}^\infty(G) \otimes \mathfrak{p}$ satisfying

$$\sum_{i=1}^l R_h f_i \otimes \text{Ad}(h)X_i = \sum_{i=1}^l f_i \otimes X_i$$

for all $h \in H$. Here $(R_h f)(x) := f(xh)$, $h \in H$, $x \in G$, $f \in \mathcal{C}^\infty(G)$. Under the G -action of $\mathcal{C}^\infty(G) \otimes \mathfrak{p}$ defined by

$$\tau_x(f \otimes X) := \tau_{x*}f \otimes X, \quad x, y \in G, f \in \mathcal{C}^\infty(G), X \in \mathfrak{p},$$

the subspace $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ is a G -submodule. The identification Ψ of $\mathcal{C}_H^\infty(G, \mathfrak{p})$ with $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ is given by

$$(8.4) \quad \Psi(f) := \sum_{i=1}^n f_i \otimes X_i, \quad f \in \mathcal{C}_H^\infty(G, \mathfrak{p}),$$

where $f(x) = \sum_{i=1}^n f_i(x) X_i$, $x \in G$, and $\{X_i\}_{i=1}^n$ is a fixed orthonormal basis of $\mathfrak{p}$ with respect to $(\cdot, \cdot)$. Then it turns out that Ψ is a G -isomorphism of $\mathcal{C}_H^\infty(G, \mathfrak{p})$ onto $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ with

$$(8.5) \quad \Psi \circ \tau_x = \tau_x \circ \Psi, \quad x \in G.$$

DEFINITION 8.3. Via Φ and Ψ , we can define a G -invariant operator $\tilde{J}$ on $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ from the Jacobi operator J_ϕ in such a way that the following diagram is commutative:

$$\begin{array}{ccccc} \Gamma(E) & \xrightarrow{\Phi^{-1}} & \mathcal{C}_H^\infty(G, \mathfrak{p}) & \xrightarrow{\Psi} & (\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H \\ \downarrow J_\phi & & & & \downarrow \tilde{J} \\ \Gamma(E) & \xrightarrow{\Phi^{-1}} & \mathcal{C}_H^\infty(G, \mathfrak{p}) & \xrightarrow{\Psi} & (\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H \end{array}$$

By (8.2), (8.3) and (8.5), the operator $\tilde{J}$ is G -invariant, that is,

$$(8.6) \quad \tilde{J} \circ \tau_x = \tau_x \circ \tilde{J}, \quad x \in G.$$

Therefore the problem of determining the spectrum of J_ϕ is reduced to doing so for the operator $\tilde{J}$ on $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$. Thus the main purpose of this section is to express the operator $\tilde{J}$ in terms of the Lie algebra $\mathfrak{g}$ of G (cf. Theorem 8.11).

8.2. For the calculus, we take a neighborhood U in G and a subset N (resp. N_K) of G (resp. K) in such a way that

- (i) $N = U \cap \exp(\mathfrak{p})$, $N_K = U \cap \exp(\mathfrak{h}_1)$,
- (ii) the map $N \times N_K \ni (y, k) \mapsto yk \in N \cdot N_K$ is a diffeomorphism,
- (iii) the projection π_K of G onto G/K is a diffeomorphism of N onto a neighborhood $\pi_K(N)$ of the origin $\{K\}$ in G/K , and
- (iv) the projection π_H of G onto G/H is a diffeomorphism of $N \cdot N_K$ onto a neighborhood $\pi_H(N \cdot N_K)$ of the origin $\{H\}$ in G/H , where $N \cdot N_K := \{yk; y \in N, k \in N_K\}$.

Now for an element $X \in \mathfrak{m} = \mathfrak{h}_1 \oplus \mathfrak{p}$, define a vector field X^* on the neighborhood $\pi_H(N \cdot N_K)$ of $\{H\}$ in G/H by

$$(8.7) \quad X_{xH}^* := \tau_{x*} X_{\{H\}} \in T_{xH} G/H, \quad x \in N \cdot N_K.$$

Similarly, for an element $X \in \mathfrak{p}$, define a vector field $\bar{X}$ on the neighborhood $\pi_K(N)$ of $\{K\}$ in G/K by

$$(8.8) \quad \bar{X}_{yK} := \tau_{y*} X_{\{K\}} \in T_{yK} G/K, \quad y \in N.$$

Let $\{X_i\}_{i=1}^m$ be an orthonormal basis of $(\mathfrak{m}, (\cdot, \cdot))$ such that $\{X_i\}_{i=1}^m$ (resp. $\{X_i\}_{i=n+1}^m$) is a basis of $\mathfrak{p}$ (resp. $\mathfrak{h}_1$). Then $\{X_i^*\}_{i=1}^m$ is an orthonormal frame field on $\pi_H(N \cdot N_K)$ such that the X_i^* , $n+1 \leq i \leq m$, are vertical and the X_i^* , $1 \leq i \leq n$, are horizontal. Also, $\{\bar{X}_i\}_{i=1}^n$ is an orthonormal frame field on $\pi_K(N)$.

REMARK. In general, the X_i^* , $1 \leq i \leq n$, are not necessarily basic vector fields. For every $f \in \mathcal{C}_H^\infty(G, \mathfrak{p})$, we can express $V = \Phi(f) \in \Gamma(E)$ as

$$V_{xH} = \sum_{i=1}^n f_i(x) \tau_{x*} X_{i\{K\}}, \quad x \in G,$$

where $f(x) = \sum_{i=1}^n f_i(x) X_i$, $x \in G$. Moreover, putting

$$(8.9) \quad \text{Ad}(k) X_i = \sum_{j=1}^n a_{ij}(k) X_j, \quad k \in K,$$

$$(8.10) \quad \tilde{f}_j(ykH) := \sum_{i=1}^n f_i(yk) a_{ij}(k), \quad y \in N, k \in N_K,$$

the section V can be expressed on the neighborhood $\pi_H(N \cdot N_K)$ as

$$(8.11) \quad V = \sum_{j=1}^n \tilde{f}_j \widetilde{\bar{X}}_j,$$

where $\tilde{f}_j$ is a function (8.10) on $\pi_H(N \cdot N_K)$ and $\widetilde{\bar{X}}_j$ is a local section of E corresponding to the vector field $\bar{X}_j$ on $\pi_H(N \cdot N_K)$ (cf. 1.1). Then we have for $X \in \mathfrak{m}$,

$$(8.12) \quad \tilde{\nabla}_{X*} V = \sum_{j=1}^n \left\{ (X^* \tilde{f}_j) \widetilde{\bar{X}}_j + \tilde{f}_j \tilde{\nabla}_{X*} \widetilde{\bar{X}}_j \right\}$$

on $\pi_H(N \cdot N_K)$. Here $(\tilde{\nabla}_{X*} \widetilde{\bar{X}}_j)_{xH}$, $x \in N \cdot N_K$, is given by

$$(8.13) \quad \left(\tilde{\nabla}_{X*} \widetilde{\bar{X}}_j \right)_{xH} = \left({}^N \nabla_W \bar{X}_j \right)_{xK},$$

where W is a locally defined vector field on G/K satisfying $W_{xK} = \phi_* X_{xH}^*$ (cf. (1.1) or (6.4)), and ${}^N \nabla$ is the Levi-Civita connection of $(G/K, g)$. This vector field W can be chosen as follows:

$$(8.14) \quad W = 0 \quad \text{for } X \in \mathfrak{h}_1,$$

$$(8.15) \quad W = \overline{(\text{Ad}(k(\cdot))X)} \quad (\text{cf. (8.8)}), \quad \text{for } X \in \mathfrak{p}.$$

In fact, since $\phi_* X_{xH}^* = 0$ for $X \in \mathfrak{h}_1$, we have (8.14). For (8.15), let $X \in \mathfrak{p}$. For a fixed point $x = y(x)k(x)$, $y(x) \in N$, $k(x) \in N_K$, we have

$$\begin{aligned} \phi_* X_{xH}^* &= \tau_{y(x)*} \tau_{k(x)*} X_{\{K\}} = \tau_{y(x)*} (\text{Ad}(k(x))X)_{\{K\}} \\ &= \overline{(\text{Ad}(k(x))X)}_{y(x)K}, \end{aligned}$$

so we can choose W as in (8.15). By (8.14), we get, for $X \in \mathfrak{h}_1$,

$$(8.16) \quad (\tilde{\nabla}_{X*} V)_{xH} = \sum_{j=1}^n (X^* \tilde{f}_j)(xH) \left(\widetilde{\bar{X}}_j \right)_{xH}.$$

By (8.15), we get in particular, for $X \in \mathfrak{p}$,

$$(8.17) \quad \left(\widetilde{{}^N \nabla_W \bar{X}_j} \right)_{\{H\}} = \left({}^N \nabla_{\bar{X}} \bar{X}_j \right)_{\{K\}}.$$

Moreover, for $X \in \mathfrak{p}$, we will show that

$$(8.18) \quad (\tilde{\nabla}_{X^*} \tilde{\nabla}_{X^*} \widetilde{X_j})_{\{H\}} = \frac{1}{4} \left([X, [X, X_j]_{\mathfrak{p}}]_{\mathfrak{p}} \right)_{\{K\}} \in T_{\{K\}} G/K,$$

where $X_{\mathfrak{p}}$ is the $\mathfrak{p}$ -component of X corresponding to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$.

First, recall the lemma below which follows from Theorems 8.1, 10.1 and 13.2 in [No], due to assumption (A).

LEMMA 8.4. *For every $Y, Z \in \mathfrak{p}$,*

$${}^N \nabla_{\bar{Z}} \bar{Y} = \frac{1}{2} \overline{([Z, Y]_{\mathfrak{p}})}, \quad \text{along the curve } \xi(t)K \text{ in } G/K$$

for a sufficiently small t such that $\xi(t) := \exp(tZ)$ belongs to N .

To establish (8.18), note that by (8.13), we have

$$(8.19) \quad (\tilde{\nabla}_{X^*} \tilde{\nabla}_{X^*} \widetilde{X_j})_{\{H\}} = ({}^N \nabla_W {}^N \nabla_W \bar{X_j})_{\{K\}},$$

where W is given by (8.15). Then for the curve $\sigma(t) := \exp(tX)K$ in G/K ,

$$\text{the right side of (8.19)} = \frac{d}{dt} {}^N P_{\sigma(t)}^{-1} ({}^N \nabla_W \bar{X_j})_{\sigma(t)} \Big|_{t=0},$$

where ${}^N P_{\sigma(t)}$ is parallel transport of $(G/K, g)$ along the curve $\sigma(t)$. Here $W_{\sigma(t)} = \bar{X}_{\sigma(t)}$, by (8.15) and $\exp(tX) \in N$, so that $k(\sigma(t)) = e$. Then we have

$$({}^N \nabla_W \bar{X_j})_{\sigma(t)} = ({}^N \nabla_{\bar{X}} \bar{X_j})_{\sigma(t)} = \frac{1}{2} \overline{([X, X_j]_{\mathfrak{p}})}_{\sigma(t)},$$

by Lemma 8.4, which also gives

$$\begin{aligned} \text{the right side of (8.19)} &= \frac{1}{2} \frac{d}{dt} {}^N P_{\sigma(t)}^{-1} \overline{([X, X_j]_{\mathfrak{p}})}_{\sigma(t)} \Big|_{t=0} \\ &= \frac{1}{2} ({}^N \nabla_{\bar{X}} \overline{[X, X_j]_{\mathfrak{p}}})_{\{K\}} = \frac{1}{4} ([X, [X, X_j]_{\mathfrak{p}}]_{\mathfrak{p}})_{\{K\}}, \end{aligned}$$

which implies (8.18).

Summing up the above, we have

LEMMA 8.5. *For $V = \Phi(f)$, $f \in \mathcal{C}_H^\infty(G, \mathfrak{p})$, we have*

(i)

$$(\tilde{\nabla}_{X^*} \tilde{\nabla}_{X^*} V)_{\{H\}} = \sum_{i=1}^n X_{\{H\}}^* (X^* \tilde{f}_j) \bar{X}_{j\{K\}}, \quad \text{for } X \in \mathfrak{h}_1,$$

(ii)

$$\begin{aligned} (\tilde{\nabla}_{X^*} \tilde{\nabla}_{X^*} V)_{\{H\}} &= \sum_{j=1}^n X_{\{H\}}^* (X^* \tilde{f}_j) \bar{X}_{j\{K\}} + (X_{\{H\}}^* \tilde{f}_j) \overline{([X, X_j]_{\mathfrak{p}})}_{\{K\}} \\ &\quad + \frac{1}{4} \tilde{f}_j(H) \overline{([X, [X, X_j]_{\mathfrak{p}}]_{\mathfrak{p}})}_{\{K\}}, \end{aligned}$$

for $X \in \mathfrak{p}$.

Our next task is to calculate $X_{\{H\}}^* \tilde{f}_j$ and $X_{\{H\}}^* X^* \tilde{f}_j$, for $X \in \mathfrak{m}$.

LEMMA 8.6. (i) For $X \in \mathfrak{h}_1$, we have

$$X_{\{H\}}^* \tilde{f}_j = Xf_j(e) + \sum_{i=1}^n f_i(e)([X, X_i], X_j),$$

and

$$\begin{aligned} X_{\{H\}}^* X^* \tilde{f}_j &= X^2 f_j(e) + 2 \sum_{i=1}^n (Xf_i)(e)([X, X_i], X_j) \\ &\quad + \sum_{i=1}^n f_i(e)([X, [X, X_i]], X_j). \end{aligned}$$

(ii) For $X \in \mathfrak{p}$, we have

$$X_{\{H\}}^* \tilde{f}_j = Xf_j(e),$$

and

$$X_{\{H\}}^* X^* \tilde{f}_j = X^2 f_j(e).$$

Proof follows immediately from the definition of $\tilde{f}_j$ (8.9), (8.10) and X^* (8.7).

LEMMA 8.7. $(\tilde{\nabla}_{\nabla_{X^*} X^*} V)_H = 0$ for all $X \in \mathfrak{m}$, and $V \in \Gamma(E)$.

PROOF. Due to assumption (A), we have $(\nabla_{X^*} X^*)_{\{H\}} = 0$ for $X \in \mathfrak{m}$, by Theorems 8.1, 13.1 in [No]. By (8.13) or (1.1), we have Lemma 8.7.

Moreover, it is known (cf. [K.N]) that under assumption (A), the curvature tensor ${}^N R$ of $(G/K, h)$ is given by

$$\begin{aligned} -({}^N R(X, Y)Z)_{\{K\}} &= \frac{1}{4}[X, [Y, Z]_{\mathfrak{p}}]_{\mathfrak{p}} - \frac{1}{4}[Y, [X, Z]_{\mathfrak{p}}]_{\mathfrak{p}} - \frac{1}{2}[[X, Y]_{\mathfrak{p}}, Z]_{\mathfrak{p}} \\ &\quad - [[X, Y]_{\mathfrak{t}}, Z], \quad X, Y, Z \in \mathfrak{p}, \end{aligned}$$

where we identify $X \in \mathfrak{p}$ with the tangent vector $X_{\{K\}} \in T_{\{K\}}G/K$. Then we get

LEMMA 8.8. For $V = \Phi(f)$, $f \in \mathcal{C}_H^\infty(G, \mathfrak{p})$, we have

$$\begin{aligned} &-({}^N R(\phi_* X^*, V)\phi_* X^*)_{\{K\}} \\ &= \begin{cases} 0, & X \in \mathfrak{h}_1, \\ \sum_{i=1}^n f_i(e) \left\{ \frac{1}{4}[X, [X_i, X]_{\mathfrak{p}}]_{\mathfrak{p}} \right. \\ \quad \left. - \frac{1}{2}[[X, X_i]_{\mathfrak{p}}, X]_{\mathfrak{p}} - [[X, X_i]_{\mathfrak{t}}, X] \right\}, & X \in \mathfrak{p}. \end{cases} \end{aligned}$$

Summing up Lemmas 8.5–8.8, we obtain

PROPOSITION 8.9. For $V = \Phi(f)$ and $f = \sum_{i=1}^n f_i X_i \in \mathcal{C}_H^\infty(G, \mathfrak{p})$, the evaluation of $J_\phi V$ at the origin $\{H\}$ in G/H is given by

$$\begin{aligned} (J_\phi V)_{\{H\}} = & - \sum_{k=1}^m \sum_{j=1}^n (X_k^2 f_j)(e) X_{j\{K\}} \\ & - \sum_{k,j=1}^n (X_k f_j)(e) [X_k, X_j]_{\mathfrak{p}\{K\}} \\ & - 2 \sum_{k=n+1}^m \sum_{j=1}^n (X_k f_j)(e) [X_k, X_j]_{\{K\}} \\ & - \sum_{k=n+1}^m \sum_{j=1}^n f_j(e) [X_k, [X_k, X_j]]_{\{K\}} \\ & - \sum_{k,j=1}^n f_j(e) [[X_k, X_j]_{\mathfrak{f}}, X_k]_{\{K\}}. \end{aligned}$$

8.3. Before we state Theorem 8.11, we need some notation.

DEFINITION 8.10. The operators D_i , $i = 0, 1, \dots, 6$, acting on $\mathcal{C}^\infty(G) \otimes \mathfrak{p}$ are given by

$$\begin{aligned} D_0 &:= \sum_{k=1}^s X_k^2 \otimes I, \\ D_1 &:= \sum_{k=1}^m X_k^2 \otimes I, \\ D_2 &:= \sum_{k=1}^n X_k \otimes P_{\mathfrak{p}} \circ \text{ad}(X_k), \\ D_3 &:= \sum_{k=n+1}^m X_k \otimes \text{ad}(X_k), \\ D_4 &:= I \otimes \sum_{k=n+1}^m \text{ad}(X_k)^2, \\ D_5 &:= I \otimes \sum_{k=1}^n \text{ad}(X_k) \circ P_{\mathfrak{f}} \circ \text{ad}(X_k), \\ D_6 &:= \sum_{k=m+1}^s X_k^2 \otimes I, \end{aligned}$$

where $P_{\mathfrak{p}}$ and $P_{\mathfrak{f}}$ are the projections of $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$ onto $\mathfrak{p}$ and $\mathfrak{f}$, respectively, $\{X_k\}_{k=1}^s$ is an orthonormal basis of $(\mathfrak{g}, (\cdot, \cdot))$ such that $\{X_i\}_{i=1}^n$ (resp. $\{X_i\}_{i=n+1}^m$, $\{X_i\}_{i=m+1}^s$) is a basis of $\mathfrak{p}$ (resp. $\mathfrak{h}_1$, $\mathfrak{h}$), I is the identity operator of $\mathcal{C}^\infty(G)$, $\mathfrak{p}$ or $\mathcal{C}^\infty(G) \otimes \mathfrak{p}$, and $(Xf)(x) := (d/dt)f(x \exp(tX))|_{t=0}$, for $X \in \mathfrak{g}$, $f \in \mathcal{C}^\infty(G)$, and $x \in G$.

It turns out that all D_i , $i = 0, 1, \dots, 6$, are independent of the choice of the above basis $\{X_k\}_{k=1}^s$ and they are G -invariant, i.e., $D_i \circ \tau_x = \tau_x \circ D_i$, for all $x \in G$. Thus since $R_h \circ Xf = \text{Ad}(h)X(R_h f)$, for $f \in \mathcal{C}^\infty(G)$, $h \in H$, and $X \in \mathfrak{g}$, all D_i keep the subspace $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ invariant. We also note that

$$(8.20) \quad D_0 = D_1 + D_6,$$

$$(8.21) \quad D_6 = I \otimes \sum_{k=m+1}^s \text{ad}(X_k)^2 \quad \text{on } (\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H,$$

by the definition of $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ and D_6 . Then by Proposition 8.9, we obtain

THEOREM 8.11. *Let ϕ be the Riemannian submersion of $(G/H, g)$ onto $(G/K, h)$ whose metrics g and h come from the $\text{Ad}(G)$ -invariant inner product $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$. Then the operator $\tilde{J}$ of $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ corresponding to the Jacobi operator J_ϕ of the submersion ϕ coincides with the operator*

$$D := -D_0 - D_2 - 2D_3 - D_4 + D_5 + D_6,$$

where all D_i are defined in Definition 8.10.

PROOF. Proposition 8.9 and (8.21) yield

$$\tilde{J}(\Psi\Phi^{-1}V)(e) = D(\Psi\Phi^{-1}V)(e),$$

for every $V \in \Gamma(E)$. For every $x \in G$, we have

$$\begin{aligned} \tilde{J}(\Psi\Phi^{-1}V)(x) &= \tau_{x^{-1}} \circ \tilde{J}(\Psi\Phi^{-1}V)(e) = \tilde{J}(\Psi\Phi^{-1}\tau_{x^{-1}*}V)(e) \\ &= D(\Psi\Phi^{-1}\tau_{x^{-1}*}V)(e) = \tau_{x^{-1}}D(\Psi\Phi^{-1}V)(e) = D(\Psi\Phi^{-1}V)(e). \quad \text{Q.E.D.} \end{aligned}$$

As applications of Theorem 8.11, we obtain

COROLLARY 8.12. *Let ϕ be the Riemannian submersion of $(G/H, g)$ onto $(G/K, h)$ whose metrics g and h come from the $\text{Ad}(G)$ -invariant inner product $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$. Assume that $(G/K, h)$ is Riemannian symmetric, $\mathfrak{g}$ is semisimple, and $(X, Y) := -F(X, Y)$, for $X, Y \in \mathfrak{g}$, where F is the Killing form of $\mathfrak{g}$.*

(i) *Then the operator $\tilde{J}$ of $(\mathcal{C}^\infty(G) \otimes \mathfrak{p})_H$ corresponding to the Jacobi operator J_ϕ of the submersion ϕ coincides with*

$$D := -D_0 - 2D_3 + 2D_6.$$

If $H = \{\text{id}\}$, then the operator $\tilde{J}$ coincides with $D := -D_0 - 2D_3$, where D_0 , D_3 and D_6 are given in Definition 8.10.

(ii) *In particular, the spectrum of the Jacobi operator J_ϕ of the Hopf fibering ϕ ;*

$$(SU(2), g) = (S^3, g) \rightarrow (SU(2)/S(U(1) \times U(1)), h) = (S^2, h)$$

is given as follows: The eigenvalues: $\frac{1}{2}l(l+1) + i$, $\frac{1}{2}l(l+1) - i$, their multiplicities: $2l+1$, where l varies over the set $\{l \in \frac{1}{2}\mathbb{Z}; l \geq 0\}$, and i varies over the set $\{l, l-1, \dots, 1-l, -l\}$. Finally, $\text{Index}(\phi) = 2$ and $\text{Nullity}(\phi) = 8$.

PROOF. (i) Since $(G/K, h)$ is symmetric, i.e., $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we have $D_2 = 0$ and $D_5 = I \otimes \sum_{k=1}^n \text{ad}(X_k)^2$. Moreover, $D_5 = -\frac{1}{2}I$ and $D_4 + D_6 = -\frac{1}{2}I$ imply (i), since $(\sum_{k=1}^n \text{ad}(X_k)^2)(X, Y) = \frac{1}{2}F(X, Y)$, and $(\sum_{k=n+1}^s \text{ad}(X_k)^2)(X, Y) = \frac{1}{2}F(X, Y)$, for $X, Y \in \mathfrak{p}$ (cf. [T.K, p. 212]). The second claim in (i) is clear because $D_6 = 0$ when $H = \{\text{id}\}$.

(ii) Let us recall the computation in [U1, §5]. In this case,

$$\begin{aligned} G &= SU(2), \\ K &= S(U(1) \times U(1)) = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}; \theta \in \mathbf{R} \right\}, \\ (X, Y) &= -4 \operatorname{Trace}(XY), \quad X, Y \in \mathfrak{g} = \mathfrak{su}(2), \\ \mathfrak{k} &= \{H_1\}_{\mathbf{R}}, \\ \mathfrak{p} &= \{U_\alpha/\sqrt{2}, V_\alpha/\sqrt{2}\}_{\mathbf{R}}, \end{aligned}$$

where

$$H_1 := \sqrt{-1}/2\sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U_\alpha := 2^{-1} \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

and

$$V_\alpha := 2^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here $\{H_1, U_\alpha/\sqrt{2}, V_\alpha/\sqrt{2}\}$ is an orthonormal basis of $(\mathfrak{g}, (\cdot, \cdot))$. We have only to know the actions of $D_3 = H_1 \otimes \operatorname{ad}(H_1)$ and $D_0 = C \otimes I$ on $\mathcal{C}^\infty(G) \otimes \mathfrak{p}$, where C is the Casimir operator $C := H_1^2 + U_\alpha^2/2 + V_\alpha^2/2$. A complete orthogonal basis of the space $\mathcal{C}^\infty(G)$ of complex valued smooth functions on G with respect to the inner product $\int_G f(x)\overline{f'(x)}dx$, $f, f' \in \mathcal{C}^\infty(G)$, with the Haar measure dx , is given as follows by the Peter-Weyl theorem. Let $D := \{l\alpha; l \in \frac{1}{2}\mathbf{Z}, l \geq 0\}$. For $\lambda = l\alpha \in D$, let (V_λ, π^λ) be the irreducible unitary representation of G with highest weight λ , and $\{v_i\}_{i=1}^{d_\lambda}$, $d_\lambda := \dim(V_\lambda)$, an orthonormal basis of V_λ with respect to the G -invariant inner product $((\cdot, \cdot))$ on V_λ . Put $\pi_{ij}^\lambda(x) := ((\pi^\lambda(x)v_i, v_j))$, $1 \leq i, j \leq d_\lambda$. Then

$$X\pi_{ij}^\lambda(x) = ((\pi^\lambda(x)\pi^\lambda(X)v_i, v_j)), \quad X \in \mathfrak{g}, 1 \leq i, j \leq d_\lambda,$$

and $\{\pi_{ij}^\lambda, \lambda \in D, 1 \leq i, j \leq d_\lambda\}$ is an orthogonal basis of $\mathcal{C}^\infty(G)$. For $\lambda = l\alpha$ with $l \in \frac{1}{2}\mathbf{Z}$, $l \geq 0$, V_λ has an orthonormal basis $\{v_m; m = l, l-1, \dots, 1-l, -l\}$ such that

$$\pi^\lambda(H_1)v_m = \sqrt{-1}mv_m/\sqrt{2},$$

for each m . Since $\pi^\lambda(C) = \frac{1}{2}l(l+1)I$ on V_λ , we get

$$H_1\pi_{ij}^\lambda(x) = (\sqrt{-1}/\sqrt{2})i\pi_{ij}^\lambda(x),$$

$$C\pi_{ij}^\lambda(x) = \frac{1}{2}l(l+1)\pi_{ij}^\lambda(x),$$

for $i, j = l, l-1, \dots, 1-l, -l$. On the other hand,

$$\operatorname{ad}(H_1)\left(\frac{U_\alpha}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{V_\alpha}{\sqrt{2}}\right), \quad \operatorname{ad}(H_1)\left(\frac{V_\alpha}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\left(\frac{U_\alpha}{\sqrt{2}}\right).$$

Thus the action of $D_3 = H_1 \otimes \text{ad}(H_1)$ on $\vartheta_\lambda \otimes \mathfrak{p}$, where $\vartheta_\lambda := \{\pi_{ij}^\lambda; 1 \leq i, j \leq d_\lambda\}_{\mathbb{C}}$, is equivalent to the matrix

$$\begin{pmatrix} \lambda_l & & & & \\ & \lambda_{l-1} & & & \\ & & \ddots & & \\ & & & \lambda_{1-l} & \\ & & & & \lambda_{-l} \end{pmatrix} \otimes \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \\ = \frac{1}{\sqrt{2}} \begin{pmatrix} \boxed{\begin{matrix} 0 & -\lambda_l \\ \lambda_l & 0 \end{matrix}} & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix} 0 & -\lambda_{-l} \\ \lambda_{-l} & 0 \end{matrix}} \end{pmatrix},$$

where $\lambda_i := (\sqrt{-1}/\sqrt{2})i$, $i = l, l-1, \dots, 1-l, -l$. Therefore the eigenvalues of D_3 on $\vartheta_\lambda \otimes \mathfrak{p}$ are given by $\pm \frac{i}{2}$, $i = l, l-1, \dots, 1-l, -l$. Hence the spectrum of $D = -D_0 - 2D_3$ is given as in (ii). Q.E.D.

Instead of the assumption of Corollary 8.12, we now assume that $K = H$. In this case, we obtain the formula for $\tilde{J}$ of the Jacobi operator J_{id} of the identity map of a normally homogeneous space $(G/H, g)$. Here we have $\mathfrak{k} = \mathfrak{h}$, $\mathfrak{h}_1 = 0$, $\mathfrak{m} = \mathfrak{p}$ and $D_3 = D_4 = 0$. Thus we obtain

COROLLARY 8.13. *Let $(G/H, g)$ be a normally homogeneous space, that is, the metric g is induced from the $\text{Ad}(G)$ -invariant inner product $(\cdot, \cdot)$ on the Lie algebra $\mathfrak{g}$. Then the operator $\tilde{J}$ of $(\mathcal{C}^\infty(G) \otimes \mathfrak{m})_H$ corresponding to the Jacobi operator J_{id} of the identity map of $(G/H, g)$ coincides with $D = -D_0 - D_2 + D_5 + D_6$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $(\cdot, \cdot)$ and D_0 , D_2 , D_5 and D_6 are given in Definition 8.10.*

In particular, assume that $(G/H, g)$ is Riemannian symmetric, $\mathfrak{g}$ is semisimple, and $(X, Y) := -F(X, Y)$ for $X, Y \in \mathfrak{g}$, where F is the Killing form of $\mathfrak{g}$. Then $D = -D_0 - I$, where I is the identity map of $(\mathcal{C}^\infty(G) \otimes \mathfrak{m})_H$.

PROOF. The last formula follows from $D_2 = 0$ and $D_5 + D_6 = -I$.

REMARK. The last formula $D = -D_0 - I$ for the Jacobi operator of the identity map of a Riemannian symmetric space was stated in [Na].

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